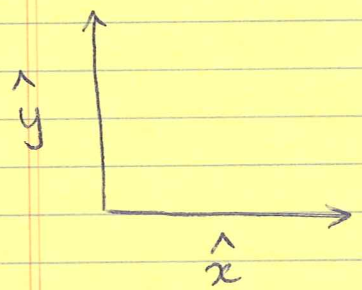
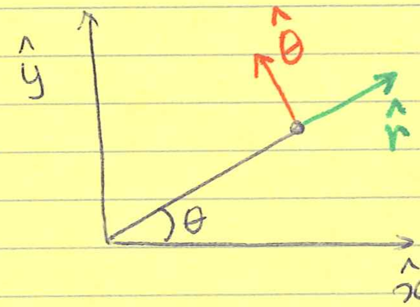
Advanced Topic

General Motion in Polar Coordinates

Consider an object undergoing circular motion
Instead of describing its position, velocity
and acceleration in cartesian co-ordinates
its much nicer to use polar co-ordinates





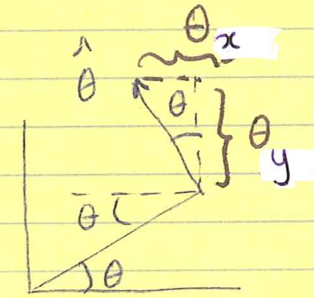
Cartesian
$(\hat{x}, \hat{y})$

Polar
$(\hat{r}, \hat{\theta})$

The two are related as follows:

$$\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$

[We can show this is a unit vector because
$|\hat{r}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$]



$$\hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$

[Once again we can show that
$|\hat{\theta}| = 1$ so it is a unit vector]

and $\hat{\theta} \cdot \hat{r} = 0 \Rightarrow$ the two vectors are orthogonal.

They are useful to define a coordinate system.

$$\boxed{\begin{aligned} \hat{r} &= \cos\theta\,\hat{x} + \sin\theta\,\hat{y} \\ \hat{\theta} &= -\sin\theta\,\hat{x} + \cos\theta\,\hat{y} \end{aligned}} \quad \text{—— (1)}$$

Now start with the object at some position $\vec{r}(t)$ at time t

$$\boxed{\vec{r}(t) = r\,\hat{r}} \quad \text{—— (2)}$$

[The time dependence is hidden in $\hat{r}$ which depends on θ and θ depends on time]

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \left(\frac{dr}{dt}\right)\hat{r} + r\,\frac{d\hat{r}}{dt}$$

Denote $\dot{r} = \dfrac{dr}{dt}$ (the dot symbol is only used for time derivatives)

Also note

$$\frac{d\hat{r}}{dt} = \frac{d}{dt}\left(\cos\theta\,\hat{x} + \sin\theta\,\hat{y}\right)$$

$$= \frac{d}{d\theta}\left(\cos\theta\,\hat{x} + \sin\theta\,\hat{y}\right) \cdot \frac{d\theta}{dt}$$

$$= \underbrace{\left(-\sin\theta\,\hat{x} + \cos\theta\,\hat{y}\right)}_{\hat{\theta}}\dot{\theta}$$

$$\boxed{\frac{d\hat{r}}{dt} = \dot{\theta}\,\hat{\theta}} \quad \text{—— (3)}$$

$$\Rightarrow \quad \vec{v}(t) = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta}$$

Just like we can decompose $\vec{v}(t)$ into its x and y components

$$\vec{v}(t) = v_x(t)\,\hat{x} + v_y(t)\,\hat{y}$$

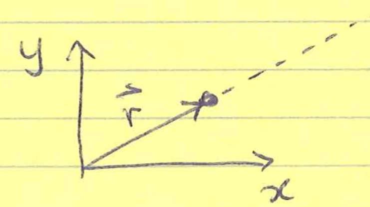
we can also decompose it into its $\hat{r}$ (radial) and $\hat{\theta}$ (angular) components.

$$\vec{v}(t) = v_r\,\hat{r} + v_\theta\,\hat{\theta}$$

$$v_r = \dot{r}$$

$$v_\theta = r\dot{\theta}$$

Special cases



a) Assume the particle is at a specific angle and θ does not change in time $\Rightarrow \dot{\theta} = 0 \quad \Rightarrow \quad v_\theta = 0$

$$v_r = \frac{dr}{dt}$$

so the particle only has radial velocity

b) assume the particle is at a constant radial distance r

$$\Rightarrow \frac{dr}{dt} = 0 \qquad \Rightarrow \quad v_r = 0$$

no radial velocity

only tangential velocity

$$v_\theta = r\dot{\theta}$$

For uniform circular motion $\dot{\theta} = \omega = \text{const.}$

and then $v_\theta = r\omega = \text{const.}$

but in general $v_\theta = r\dot{\theta}$ and will depend on time.

In general

$$\boxed{\begin{aligned} \vec{v}(t) &= \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta} \\ &\equiv v_r\,\hat{r} + v_\theta\,\hat{\theta} \end{aligned}} \qquad \text{— (4)}$$

where $\left.\begin{aligned} \dot{r} &= \frac{dr}{dt} \\ \dot{\theta} &= \frac{d\theta}{dt} \end{aligned}\right\}$ in general both r and θ can change in time

Now let us calculate the acceleration:

Start with Eq. (4)

$$\vec{v}(t) = \dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta}$$

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \underbrace{\frac{d}{dt}(\dot{r}\,\hat{r})}_{\text{term 1}} + \underbrace{\frac{d}{dt}(r\,\dot{\theta}\,\hat{\theta})}_{\text{term 2}}$$

We will use the chain rule and product rule.

$$\frac{d}{dt}(\underbrace{f\,g}_{\text{two functions}}) = \left(\frac{df}{dt}\right) g + f\left(\frac{dg}{dt}\right)$$

$$\frac{d}{dt}(\dot{r}\,\hat{r}) = \left(\frac{d}{dt}\dot{r}\right)\hat{r} + \dot{r}\,\frac{d\hat{r}}{dt}$$

Denote $\quad \frac{d}{dt}\dot{r} = \ddot{r}$

From Eq. (3) $\quad \frac{d\hat{r}}{dt} = \dot{\theta}\,\hat{\theta}$

$$\Rightarrow \frac{d}{dt}(\dot{r}\,\hat{r}) = \ddot{r}\,\hat{r} + \dot{r}\,\dot{\theta}\,\hat{\theta} \qquad \text{—— (A)}$$

(term 1)

$$\frac{d}{dt}(r\dot{\theta}\hat{\theta}) = \left(\frac{dr}{dt}\right)\dot{\theta}\hat{\theta} + r\left(\frac{d}{dt}\dot{\theta}\right)\hat{\theta}$$

$$+ r\dot{\theta}\frac{d}{dt}\hat{\theta}$$

$$= \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}$$

Now from Eq. (1)

we have $\hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$

$$\Rightarrow \frac{d\hat{\theta}}{dt} = \frac{d}{d\theta}\left[-\sin\theta\,\hat{x} + \cos\theta\,\hat{y}\right]\underbrace{\left(\frac{d\theta}{dt}\right)}_{=\dot{\theta}}$$

$$= \left(-\cos\theta\,\hat{x} - \sin\theta\,\hat{y}\right)\dot{\theta}$$

From Eq.(1)

$$= -\hat{r}\dot{\theta}$$

$$\boxed{\frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{r}}$$

$\Rightarrow$ Second term

$$\frac{d}{dt}(r\dot{\theta}\hat{\theta}) = \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta}$$

$$+ r\dot{\theta}(-\dot{\theta}\hat{r})$$

——— B

Putting term 1 and term 2 together Eqs. A and B gives the full acceleration

$$\vec{a}(t) = \underbrace{\frac{d}{dt}(\dot{r}\hat{r})} + \underbrace{\frac{d}{dt}(r\dot{\theta}\hat{\theta})}$$

$$= \left(\ddot{r}\,\hat{r} + \dot{r}\,\dot{\theta}\,\hat{\theta}\right) + \left(\dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\,\hat{\theta} - r(\dot{\theta})^2\,\hat{r}\right)$$

$$\vec{a}(t) = \ddot{r}\,\hat{r} - r(\dot{\theta})^2\,\hat{r} + 2\dot{r}\dot{\theta}\,\hat{\theta} + r\ddot{\theta}\,\hat{\theta}$$

$$\boxed{\vec{a}(t) = \left[\ddot{r} - \underbrace{r(\dot{\theta})^2}\right]\hat{r} + \left(\underbrace{2\dot{r}\dot{\theta}} + r\ddot{\theta}\right)\hat{\theta}}$$

linear acceleration in the radial direction due to a change of the radial speed.

CENTRIPETAL ACCELERATION

CORIOLIS ACCELERATION (when both r and θ change with time)

linear acceleration in the tangential direction due to a change in the magnitude of angular velocity