Time Dilation

Consider a clock at rest in S' frame and consider two events A and B occurring at the same point x_0'

A : x_0' t_A'

B : x_0' t_B'

The interval $\tau = t_B' - t_A'$ is the time interval in the rest frame. It is called the proper time interval.

To find the time interval in the S frame we use

$$t = \gamma\left(t' + \frac{u}{c^2} x'\right)$$

$$t_A = \gamma\left(t_A' + \frac{u}{c^2} x_0'\right)$$

$$\Rightarrow t_B - t_A = T$$

$$t_B = \gamma\left(t_B' + \frac{u}{c^2} x_0'\right)$$

$$= \gamma(t_B' - t_A') = \gamma\tau$$

$$\Rightarrow T = \gamma\tau = T = \frac{1}{\sqrt{1 - u^2/c^2}}\tau$$

The time interval in the moving frame S' is greater than that in the rest frame

$\Rightarrow$ Moving clocks run slow

This effect is called time dilation

frequency in rest frame $\nu_0 = \frac{1}{\tau_0}$

frequency in moving frame

$$\nu = \frac{1}{\text{time period}} = \frac{1}{\gamma(1-\beta)}\nu_0$$

$$= \frac{\sqrt{(1-\beta)(1+\beta)}}{1-\beta}\nu_0$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{(1-\beta)(1+\beta)}}$$

$$\boxed{\nu = \sqrt{\frac{1+\beta}{1-\beta}}\,\nu_0}$$

ν is the frequency in the ground frame S (or the observer's rest frame)

ν_0 is the frequency in the rest frame of the source, S'

and u is the relative speed of source and observer.

$\nu > \nu_0$ if the source is coming toward the observer the frequency is increased.

$\Rightarrow$ light is blue shifted

Example 1

One of the most prominent spectral lines of hydrogen is the H_α line, a bright red line with a wavelength of 656.1 nm.

What is the expected wavelength of the H_α line from a star receding with a speed 3000 km/s?

Since the star is receding $\beta \to -\beta$

$$\nu = \nu_0 \sqrt{\frac{1-\beta}{1+\beta}}$$

frequency is lower or wavelength is longer $\lambda = \frac{c}{\nu}$

$\Rightarrow$ red shifted

$$\frac{c}{\lambda} = \frac{c}{\lambda_0}\sqrt{\frac{1-\beta}{1+\beta}} \Rightarrow \lambda = \lambda_0 \sqrt{\frac{1+\beta}{1-\beta}}$$

$$\beta = \frac{u}{c} = \frac{3 \times 10^6 \text{ m/s}}{3 \times 10^8 \text{ m/s}} = 10^{-2}$$

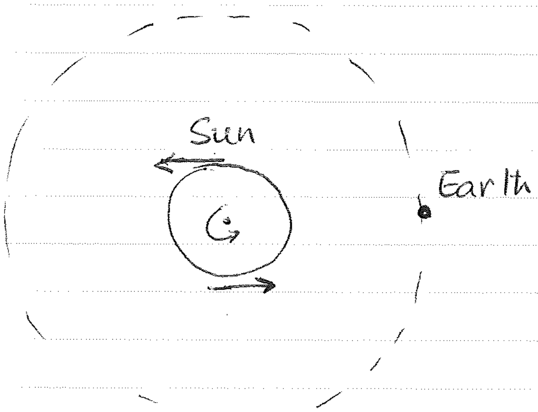
$$\Rightarrow \lambda = (656.1)\sqrt{\frac{1.01}{0.99}} = 662.7 \text{ nm}$$

Spectral line shift $\Delta\lambda = 6.6$ nm

This is how the speeds of stars and galaxies are deduced $\rightarrow$ by measuring red shifts & blue shifts.

Example 2

The H_α line measured on earth from opposite ends of the sun's equator differ in wavelength by 9 pico m $= 9 \times 10^{-12}$ m. Assume that the effect is caused by rotation of the sun find the period of rotation of the sun. The diameter of the sun is 1.4×10^{6} km.



$$\Delta\lambda = \lambda_0\left(\sqrt{\frac{1+\beta}{1-\beta}} - \sqrt{\frac{1-\beta}{1+\beta}}\right)$$

$$= \lambda_0\left(\frac{1+\beta}{\sqrt{1-\beta^2}} - \frac{1-\beta}{\sqrt{1-\beta^2}}\right)$$

$$\boxed{\Delta\lambda = \frac{2\beta}{\sqrt{1-\beta^2}}\lambda_0}$$

$$\underbrace{\left(\frac{\Delta\lambda}{\lambda_0}\right)^2}_{\equiv a} = \frac{4\beta^2}{1-\beta^2} \Rightarrow a(1-\beta^2) = 4\beta^2$$

$a \ll 1$

$$\Rightarrow a = (4+a)\beta^2$$

$$\Rightarrow \beta^2 = \frac{a}{4+a}$$

$$\beta^2 = \frac{a}{4(1+a/4)} \simeq \frac{a}{4}\left(1+\frac{a}{4}\right)^{-1}$$

binomial expansion

$$\beta^2 = \frac{a}{4+a}$$

$$\left(1+\frac{a}{4}\right)^{-1} \simeq 1-\frac{a}{4}+\frac{a^2}{2\times16}$$

$a \ll 1$

$$\beta = \frac{(\Delta\lambda/\lambda_0)}{\sqrt{4+\frac{\Delta\lambda}{\lambda_0}}} =$$

$$\beta^2 = \frac{a}{4}\left(1-\frac{a}{4}+\frac{a^2}{32}\right)$$

$$= \frac{a}{4} - \frac{a^2}{16} + \frac{a^2}{4\times32}$$

to lowest order

$$\beta^2 \simeq a/4 \Rightarrow \beta = \frac{\sqrt{a}}{2} = \frac{\Delta\lambda}{2\lambda_0}$$

$$\beta = \frac{9\times10^{-12}\ \mathrm{m}}{2\times656.1\times10^{-9}\ \mathrm{m}} = 6.86\times10^{-6}$$

$$v = 2057.6\ \mathrm{m/s}$$

$$T = \frac{2\pi R_s}{v} = \frac{\pi\times1.4\times10^{6}\times10^{3}\ \mathrm{m}}{2057\ \mathrm{m/s}} \times \frac{1}{3600} =$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$