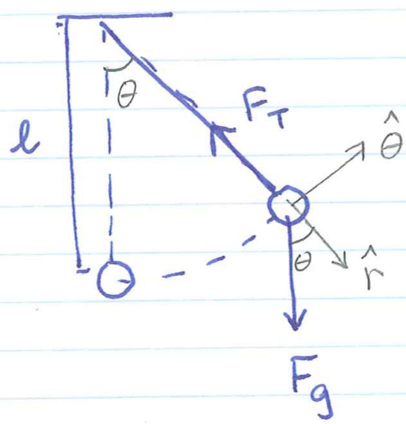
Pendulum : General Motion



Along the radial direction forces balance

$$F_T = F_g \cos\theta$$

Along the tangential direction there is a net force which causes acceleration

$$-F_g \sin\theta \,\hat{\theta} = m\, a_\theta \,\hat{\theta}$$

$$= m\, l\, \ddot{\theta}\, \hat{\theta}$$

$$\Rightarrow \quad m l \ddot{\theta} + mg \sin\theta = 0$$

$$\Rightarrow \quad \ddot{\theta} + g/l \sin\theta = 0$$

Define $\boxed{g/l = \omega_0^2}$

$$\Rightarrow \quad \boxed{\ddot{\theta} + \omega_0^2 \sin\theta = 0}$$

This is a general differential equation valid for any θ.

Harmonic Oscillator

Case I Small angle oscillations

$$\sin\theta \simeq \theta$$

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

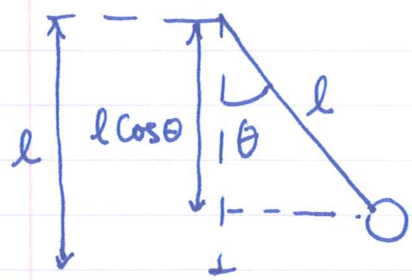
has a solution $\theta(t) = A\cos(\omega_0 t)$

$$\omega_0 = \sqrt{g/l} \qquad A = \theta_{max}$$

This is the case we have discussed in class and in the exam.

Case II   Pendulum beyond harmonic approximation

It can be solved either as an equation of motion which requires solving the differential equation or by energy conservation using integration.



$$E = \frac{1}{2} m \left(\frac{d\theta}{dt}\right)^2 l^2 + mg(l - l\cos\theta)$$

$$\boxed{E = \frac{1}{2} m l^2 \left(\frac{d\theta}{dt}\right)^2 + mgl(1-\cos\theta)}$$

E is conserved. Solve for $\left(\frac{d\theta}{dt}\right)$

$$\Rightarrow \left(\frac{d\theta}{dt}\right)^2 = \frac{2E}{ml^2} - \frac{2g}{l}(1-\cos\theta)$$

Now $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= 1 - 2\sin^2\theta$

$\Rightarrow 2\sin^2\theta = (1-\cos 2\theta)$

$\Rightarrow \boxed{2\sin^2 \theta/2 = 1-\cos\theta}$

$$\boxed{\left(\frac{d\theta}{dt}\right)^2 = \frac{2E}{ml^2} - 4\left(\frac{g}{l}\right)\sin^2 \theta/2}$$

$$\boxed{\begin{aligned} \Rightarrow \quad \frac{d\theta}{dt} &= \pm \sqrt{\frac{2E}{ml^2} - 4\left(\frac{g}{l}\right) \sin^2 \theta/2} \\[2em] \Rightarrow \quad t &= \int \frac{d\theta}{\sqrt{\frac{2E}{ml^2} - 4 \frac{g}{l} \sin^2 \theta/2}} \end{aligned}}$$

Define $\omega_0^2 = g/l$

$$\frac{2E}{ml^2} - 4\omega_0^2 \sin^2(\theta/2) = \omega_0^2 \left[\frac{2E}{ml^2 \omega_0^2} - 4 \sin^2 \theta/2 \right]$$

Now $\underbrace{[ml^2 \omega_0^2]}_{\text{has the right units of energy}} = [E]$

For this problem the natural unit for angular frequency is $\omega_0 = \sqrt{g/l}$ and the natural unit for E is $ml^2\omega_0^2$

Measure E in these units

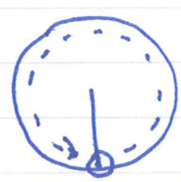
Define $\epsilon = \dfrac{E}{\cancel{2} ml^2 \omega_0^2}$

$$\frac{d\theta}{dt} = \pm \omega_0 \sqrt{2\epsilon - 4\sin^2 \theta/2}$$

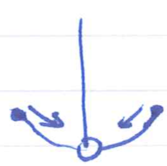
$$t = \frac{1}{\omega_0} \int \frac{d\theta}{\sqrt{2\epsilon - 4\sin^2\theta/2}}$$

<u>Three regimes</u>:

$\epsilon > 2$ pendulum revolves around



$\epsilon < 2$ pendulum oscillates



$\epsilon = 2$ Fixed point $\rightarrow$ it remains

stuck at the top

unstable fixed point

$\underline{\epsilon > 2}$

Time for 1 revolution

$$T_{rev} = \int_0^{2\pi} \frac{d\theta}{\sqrt{2\epsilon - 4\sin^2\theta/2}} = 2\int_0^{\pi} \frac{d\theta}{\sqrt{2\epsilon - 4\sin^2\theta/2}}$$

$$= \frac{8}{\epsilon - 2} K\left(\frac{2}{2-\epsilon}\right)$$

where $K(m)$ is the complete elliptic integral of the first kind.

$\underline{\epsilon < 2}$

θ_{max} is obtained by setting

$$\frac{d\theta}{dt} = \pm\sqrt{2\epsilon - 4\sin^2\theta/2} = 0$$

$$\Rightarrow \quad \epsilon = 2\sin^2\frac{\theta_{max}}{2} \Rightarrow \boxed{\theta_{max} = 2\sin^{-1}\left(\frac{\epsilon}{2}\right)}$$

Time period of oscillation

$$T_{osc} = 2\int_{-\theta_{max}}^{\theta_{max}} \frac{d\theta}{\sqrt{2\epsilon - 4\sin^2\theta/2}} = 4\int_0^{\theta_{max}} \frac{d\theta}{\sqrt{2\epsilon - 4\sin^2\theta/2}}$$

$$= 4K\left(\frac{\epsilon}{2}\right)$$

Use mathematica

$$\text{Trev}[e\_] := 2\ \text{NIntegrate}\left[\frac{1}{\sqrt{2e - 4\,\text{Sin}\left[\frac{\theta}{2}\right]^2}}, \{\theta, 0, \pi\}\right];$$

$$\text{Tosc}[e\_] := 4\ \text{NIntegrate}\left[\frac{1}{\sqrt{2e - 4\,\text{Sin}\left[\frac{\theta}{2}\right]^2}}, \{\theta, 0, 2\,\text{ArcSin}\left[\frac{e}{2}\right]\}\right];$$