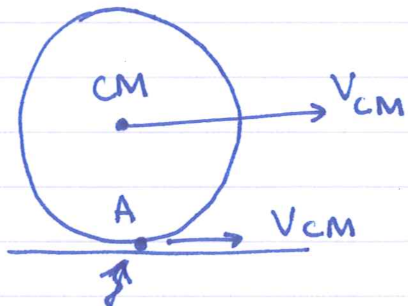
Translation & Rotation

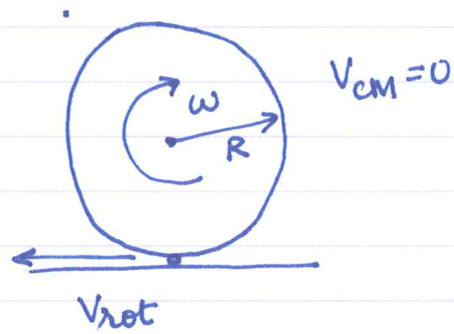
$$K = K^{CM} + K^{rot}$$

$$\boxed{K = \frac{1}{2} m V_{CM}^2 + \frac{1}{2} I \omega^2} \quad \text{—①}$$

Only translation

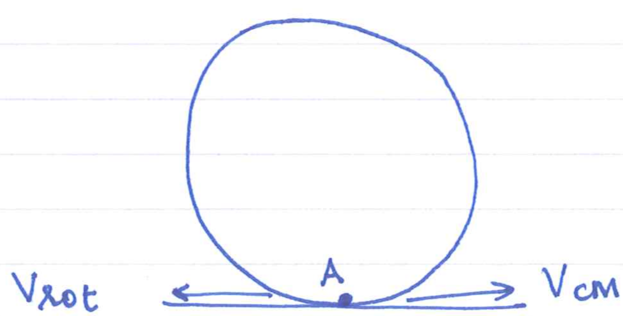


Only rotation



$$V_{rot} = R\omega$$

Rotation & Translation



$$\vec{V}_A = \vec{V}_{CM} + \vec{V}_{rot}$$

$$|\vec{V}_A| = |V_{CM} - R\omega|$$

IF ball rolls without slipping $\Rightarrow \vec{V}_A = 0$

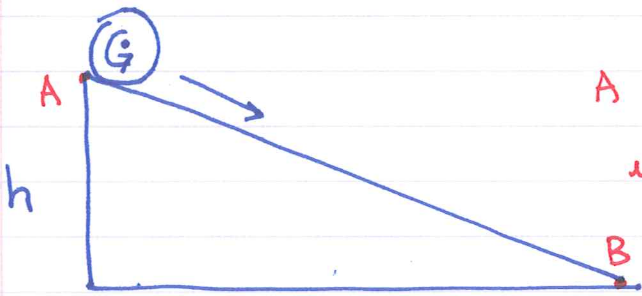
the relative velocity of ball with respect to ground at A is zero.

$$\Rightarrow V_{CM} = R\omega \Rightarrow \boxed{\omega = \frac{V_{CM}}{R}} \quad \text{—②}$$

Substitute (2) in (1)

$$K = \frac{1}{2} m V_{CM}^2 + \frac{1}{2} I \left(\frac{V_{CM}}{R}\right)^2$$

$$= \frac{1}{2} m V_{CM}^2 \left[1 + \frac{I}{m R^2} \right]$$



A ball rolls down an incline plane

What is its velocity when it reaches the bottom?

Use total energy conservation:

$$E_A = E_B$$

$$\underbrace{\cancel{K_A}}_{0} + V_A = K_B + \underbrace{\cancel{V_B}}_{0}$$

$$mgh = K_B^{CM} + K_B^{rot} = \frac{1}{2} m V_{cm}^2 \left[1 + \frac{I}{m R^2} \right]$$

$$\Rightarrow \boxed{V_{CM}^2 = \frac{2gh}{\left(1 + \frac{I}{m R^2}\right)}}$$

Moment of Inertia of different Shapes

	I	$I/MR^2 \equiv \alpha$
Hoop	MR^2	1
Disk	$\frac{MR^2}{2}$	$\frac{1}{2}$
Hollow Ball/ Spherical Shell	$\frac{2}{3}MR^2$	$\frac{2}{3}$
Solid Ball / Sphere	$\frac{2}{5}MR^2$	$\frac{2}{5}$

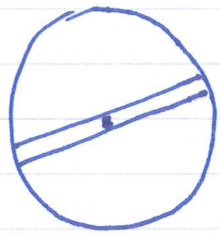
$$\boxed{v_{cm}^2 = \frac{2gh}{1+\alpha}}$$

Check: For $\alpha = 0$ i.e. for $I = 0$ or no rotation

$$v_{cm}^2 = 2gh \quad ✓$$

Analyzing the race

(1)



Hoop with a rod across the diameter

M_h = mass of hoop

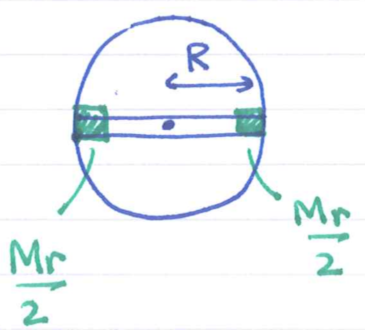
M_r = mass of rod

R = radius of hoop

$$I_1 = I_{hoop} + I_{rod}$$

$$= M_h R^2 + \frac{M_r (2R)^2}{12} = \left(M_h R^2 + \frac{M_r R^2}{3}\right)$$

(2)

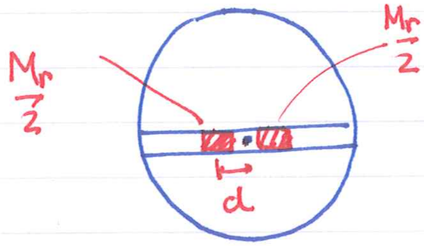


Hoop with rod but with masses concentrated at the edges

$$I_2 = M_h R^2 + \frac{M_r}{2} R^2 \times 2$$ from the two masses

$$I_2 = M_h R^2 + M_r R^2$$

③



Hoop with masses concentrated at a distance d away from the center

$d \ll R$

(We don't want to make $d=0$ otherwise the moment of inertia will be zero)

$$I_3 = M_h R^2 + \frac{M_r}{2} d^2 \times 2$$

$$= M_h R^2 + M_r d^2$$

$d \ll R.$

$$I_3 < I_1 < I_2$$

∴ Which of these objects will reach the ground fastest?

The larger the moment of inertia I the more the rotational kinetic energy K^{rot} of that object. Since $K^{rot} + K^{cm}$ must add up to the same potential energy due to gravity mgh we see that a larger K^{rot} means a smaller K^{cm}

⇒ $\boxed{\text{the object with the larger I has a small } V_{cm}}$ & will take longer to reach the ground.

$$V_{CM}^2 = \frac{2gh}{1 + \frac{I}{mR^2}}$$

$m = M_h + M_r$ total mass of object.

and I has been calculated for the three mass distributions so we can get V_{cm} in all cases.