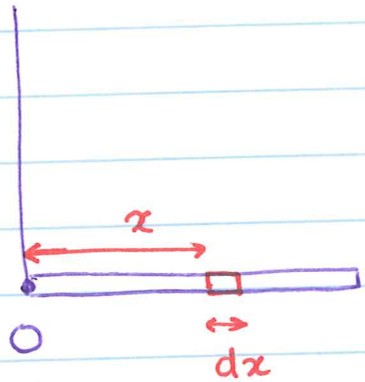
# Moment of Inertia of different Shapes

1 


Rod of Mass M
Length L
anchored at the edge

Break up the rod into small portions of size dx.

$$\text{Density} = \frac{\text{mass}}{\text{length}} \Rightarrow \rho = \frac{M}{L} \qquad \text{— (1.1)}$$

$dI$ = moment of inertia of small portion dx

$$= \text{mass of portion} \times \left(\begin{array}{l}\text{distance from axis} \\ \text{of rotation}\end{array}\right)^2$$

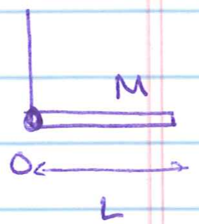
$$= \underbrace{\rho\, dx}_{\substack{\frac{\text{Mass}}{\text{length}} \times \text{length} \\ \text{dimensions of mass}}} \times (x^2)$$

$$I = \int dI = \rho \int_0^L x^2\, dx = \rho \left.\frac{x^3}{3}\right|_0^L = \frac{\rho L}{3}$$

— (1.2)

Substitute for $\rho$ from Eq. (1.1)

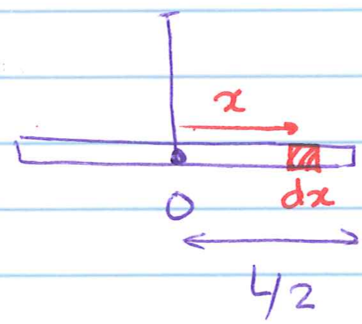
$$I = \frac{M}{L}\frac{L^3}{3} = \frac{1}{3} M L^2$$




$$\boxed{I_{rod} = \frac{ML^2}{3}}$$

about an axis through the edge of the rod.

2




Mass of rod M

Length L

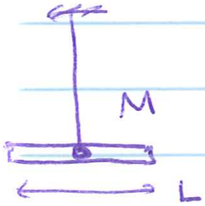
axis of rotation through center of rod.

$$dI = \rho dx \cdot x^2$$

$$I = \left[\rho \int_0^{L/2} x^2\, dx\right] \times 2$$

Note upper limit goes only till $L/2$ and the multiplication by 2 accounts for both left and right sides.

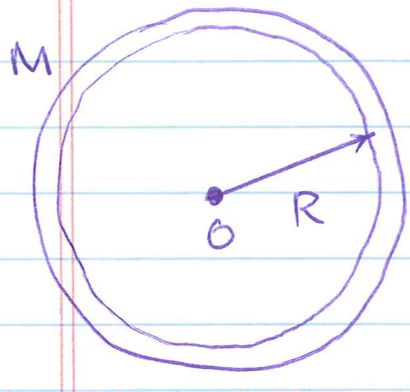
$$I = 2\rho \left.\frac{x^3}{3}\right|_0^{L/2} = \frac{2\rho}{3}\left(\frac{L}{2}\right)^3 = \frac{2}{3}\frac{M}{L}\frac{L^3}{8}$$




$$\boxed{I_{rod} = \frac{ML^2}{12}}$$

3 Hoop (thin hoop – neglect thickness)

M

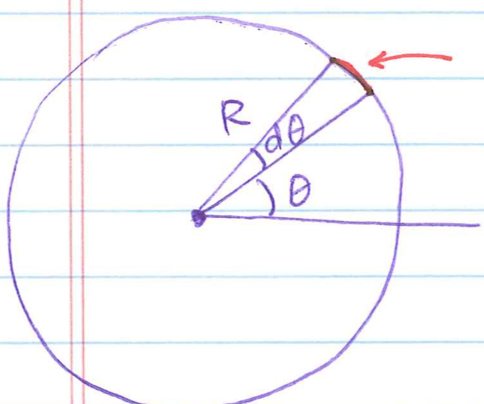



Axis of rotation through center of hoop

Since all the points on the hoop are the same distance away, R, from O

$$\boxed{I_{hoop} = MR^2}$$

Can also show this by integration




arc length = $R\,d\theta$

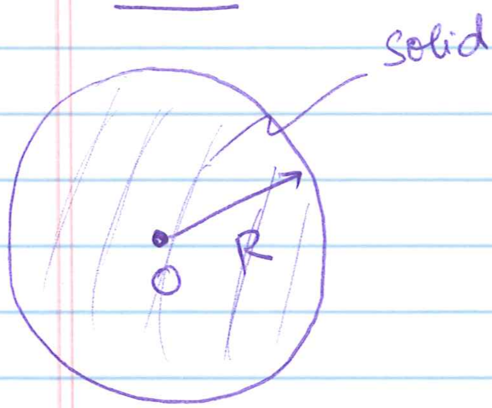
$$\frac{\text{Mass}}{\text{Length}} = \rho = \frac{M}{2\pi R}$$

$$dI = \underbrace{(\rho R\,d\theta)}_{\text{mass of arc length}} R^2$$

$$I = \int dI = \rho R^3 \int_0^{2\pi} d\theta = \frac{M}{2\pi R} R^3\, 2\pi = MR^2$$

constant for all theta
so it can be pulled out of the integral

## 4 Disk

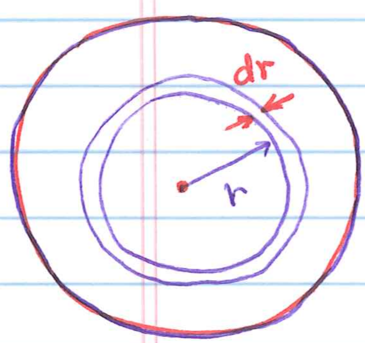



Mass M

Radius R

Axis of rotation through O

Break up the disk into thin hoops.
The hoops have different radii.




Consider a hoop at a radius r with thickness dr

$$\rho = \frac{\text{Mass}}{\text{Area}} = \frac{M}{\pi R^2}$$

$$I_{hoop} = (dm)\, r^2 = (\rho \times \text{area of hoop})\, r^2$$

$$= \rho\,(2\pi r\, dr)\, r^2$$

$$I_{disk} = \int I_{hoop} = \int_0^R \rho\,(2\pi r\, dr)\, r^2$$

$$= 2\pi\rho \int_0^R r^3\, dr = 2\pi\rho\, \frac{R^4}{4}$$

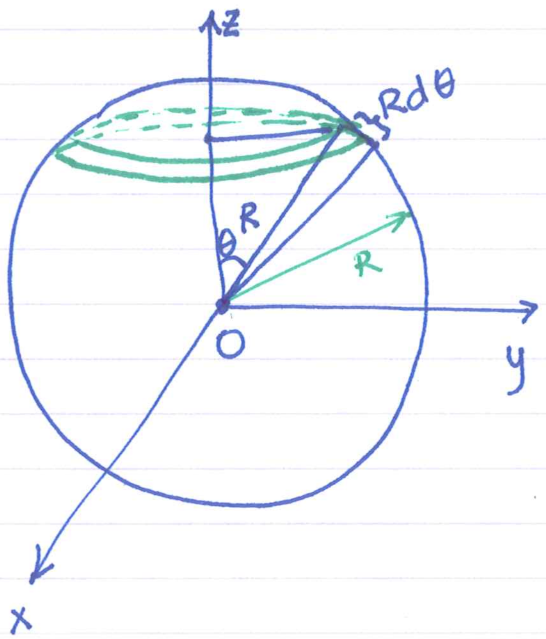
$$= 2\pi\, \frac{M}{\pi R^2}\, \frac{R^4}{4} = \frac{MR^2}{2} \Rightarrow \boxed{I_{disk} = \frac{MR^2}{2}}$$

## (5) Spherical Hollow Shell:

Axis of rotation along $\hat{z}$

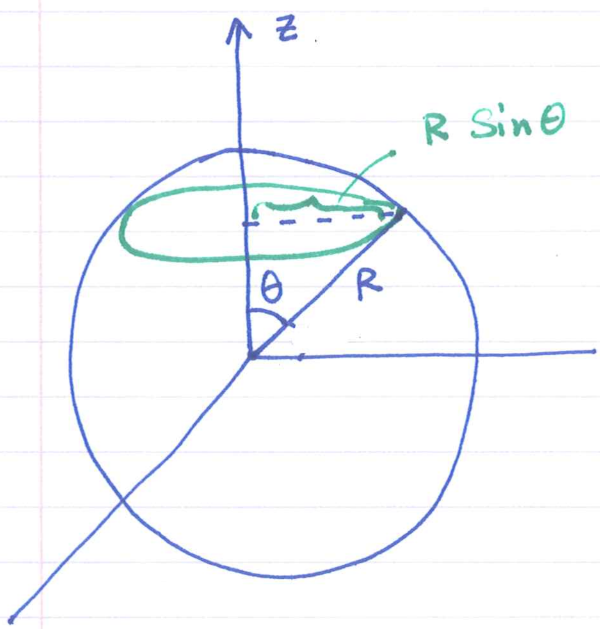
Mass M

Radius R

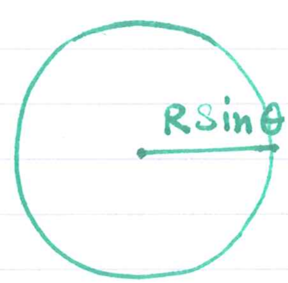



Slice the spherical shell into hoops perpendicular to the z-axis.

The radius of the hoop varies from zero at the poles to R at the equator




Top view of hoop




Radius of hoop $= R\sin\theta$

$\theta = 0$ at N pole

$\theta = \frac{\pi}{2}$ at equator

$\theta = \pi$ at S pole.

$$\rho = \frac{\text{mass}}{\text{area}} = \frac{M}{\underbrace{4\pi R^2}_{\text{surface area of spherical shell}}}$$

$$I_{hoop} = (dm)\, r^2$$

$$= \rho\,(\text{area of hoop})\, r^2$$

$$= \rho\,(2\pi\, R\sin\theta \cdot R\, d\theta)\,(R\sin\theta)^2$$

$$= 2\pi\, \rho\, R^4 \sin^3\theta\, d\theta$$

$$I_{shell} = \int I_{hoop} = 2\pi\,\rho\, R^4 \underbrace{\int_0^{\pi} \sin^3\theta\, d\theta}_{\frac{4}{3}}$$

$$= 2\pi\, \frac{M}{4\pi R^2}\, R^4\, \frac{4}{3}$$

$$\boxed{I_{shell} = \frac{2}{3} M R^2}$$

Integral of $\int_0^{\pi} \sin^3\theta \, d\theta$

$$= \int_0^{\pi} \underbrace{\sin^2\theta}_{1-\cos^2\theta} \underbrace{\sin\theta \, d\theta}_{-d\cos\theta}$$

Let $\cos\theta = x$

$$= -\int_1^{-1} (1-x^2) \, dx$$

use negative sign to switch limits.

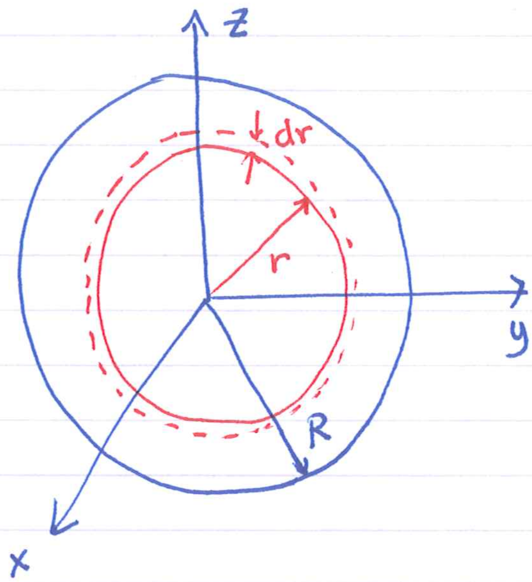
$$= \left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1}$$

$$= 1 - \frac{1}{3} - \left[-1 + \frac{1}{3}\right]$$

$$= \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

$$\boxed{\int_0^{\pi} \sin^3\theta \, d\theta = \frac{4}{3}}$$

## ⑥ Solid Sphere




Axis of rotation along $\hat{z}$

Mass $M$

Radius $R$

There are several ways of slicing and dicing the sphere.

We could slice it into disks.

The algebra is simpler if we decompose the sphere into shells like an onion.

$$\rho = \frac{\text{Mass}}{\text{Volume}} = \frac{M}{\frac{4}{3}\pi R^3}$$

$$I_{\text{shell}} = \frac{2}{3}(dm)\, r^2 = \frac{2}{3}\rho\,(4\pi r^2\, dr)\, r^2$$

$$I_{\text{sphere}} = \frac{2}{3}\,\frac{M}{\frac{4}{3}\pi R^3}\, 4\pi \underbrace{\int_0^R r^4\, dr}_{R^5/5}$$

$$\boxed{I_{\text{sphere}} = \frac{2}{5} M R^2}$$