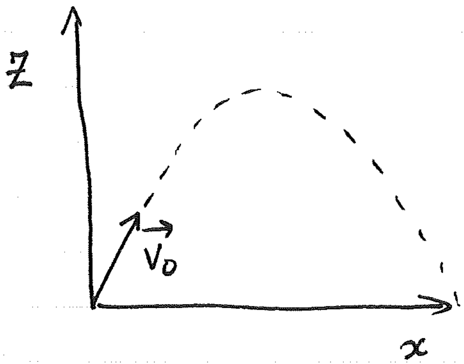
N10 PROJECTILE MOTION

Motion in a uniform gravitational



$$\vec{a} = -g\hat{z}$$

$$\frac{d\vec{V}}{dt} = \vec{a}$$

$$\Rightarrow d\vec{V} = \vec{a}\,dt$$

$$\Rightarrow \int_{t=0}^{t} d\vec{V} = \int_{0}^{t} \vec{a}\,dt = \vec{a}\int_{0}^{t} dt$$

constant in time
so $\vec{a}$ can be taken
out of the integral.

$$\Rightarrow \vec{V}(t) - \underbrace{\vec{V}(0)}_{\equiv \vec{V}_0} = \vec{a}\,t$$

$$\Rightarrow \boxed{\vec{V}(t) = \vec{V}_0 + \vec{a}\,t} \Rightarrow \begin{cases} V_x(t) = V_{0x} \\ V_z(t) = V_{0z} - gt \end{cases}$$

$$\vec{V}(t) = \frac{d\vec{r}}{dt}$$

$$\Rightarrow \int_{t=0}^{t} d\vec{r} = \int_{0}^{t} \vec{V}(t)\,dt$$

$$\Rightarrow \quad \vec{r}(t) - \vec{r}(0) = \int_0^t (\vec{V}_0 + \vec{a}\, t)\, dt$$

$$= \vec{V}_0 \int_0^t dt + \vec{a} \int_0^t t\, dt$$

constant in time

$$\vec{r}(t) \underbrace{-\vec{r}(0)}_{\equiv \vec{r}_0} = \vec{V}_0 t + \frac{1}{2} \vec{a}\, t^2$$

$$\Rightarrow \quad \boxed{\vec{r}(t) = \vec{r}_0 + \vec{V}_0 t + \frac{1}{2} \vec{a} t^2}$$

$$\Rightarrow \begin{cases} x(t) = x_0 + V_{0x} t \\ z(t) = z_0 + V_{0z} t - \frac{1}{2} g t^2 \end{cases}$$

We can eliminate time from the two equations and get the trajectory $z(x)$

$$z = z_0 + V_{0z} \left(\frac{x - x_0}{V_{0x}} \right) - \frac{1}{2} g \frac{(x - x_0)^2}{V_{0x}^2}$$

Without loss of generality choose $x_0 = z_0 = 0$.

$$\boxed{z = \frac{V_{0z}}{V_{0x}} x - \frac{g}{2 V_{0x}^2} x^2}$$

Drag & Terminal speed

The drag force on an object moving through a medium at a speed v is given by

$$F_D = \frac{1}{2} \underbrace{C}_{\text{drag coefficient}} \underbrace{\rho}_{\text{density of medium}} \underbrace{A}_{\text{object's cross sectional area}} v^2$$

Check units:

$$C = \#$$

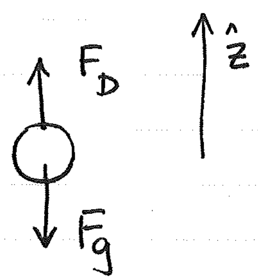
$$C \rho A v^2 = \# \left[\frac{M}{L^3} \cdot L^2 \cdot \frac{L^2}{T^2} \right] = \frac{M L}{T^2} = [\text{Force}] \checkmark$$

FREE FALL

If the only force on an object is its weight $\vec{F}_g = m\vec{g}$ it is said to be in "free fall".

TERMINAL SPEED

Motion of an object due to gravity & drag



$$F_D - F_g = m a_z$$

$$m a_z = \frac{1}{2} C \rho A V_z^2 - mg$$

$$a_z = -g \left(1 - \frac{1}{2} \frac{C \rho A}{mg} V_z^2 \right)$$

Now the term $\frac{1}{2} C \frac{\rho A}{m}$ must have dimensions of $(\text{velocity})^{-2}$. Define it to be

$$V_T^2 = \frac{2 mg}{C \rho A}$$

$$\boxed{V_T = \sqrt{\frac{2mg}{C\rho A}}}$$

$$\boxed{a_z = -g \left(1 - \frac{V_z^2}{V_T^2} \right)}$$

Note: The drag reduces the downward acceleration. When the velocity of the object $V_z = V_T$ the acceleration becomes zero. At this point $F_D = F_g$ and the object moves with a constant speed V_T called the terminal speed.

$$V_T = \sqrt{\frac{2mg}{C\rho A}}$$

is the terminal speed with which an object moves in a medium when the force of gravity is exactly balanced by the drag force of air.

Check units: $\underbrace{M}_{m} \underbrace{\left(\frac{L}{T^2}\right)}_{g} \underbrace{\left(\frac{L^3}{M}\right)}_{1/\rho} \underbrace{\frac{1}{L^2}}_{1/A} = \frac{L^2}{T^2} = (\text{vel})^2$ ✓

In the equation

$$a_z = -g\left(1 - \frac{V_z^2}{V_T^2}\right)$$

if we want to solve for $V_z(t)$ it is complicated by the fact that the acceleration itself depends on $V_z(t)$.

$$\boxed{\frac{dV_z}{dt} - \frac{g}{V_T^2} V_z^2 = -g}$$

This differential equation describes the full solution for $V_z(t)$

We can make some simple estimates in certain limits:

Consider that the object starts from rest so that in the early times V_z is small (compared to V_T) and we can neglect the second term which is of order $\Theta(V_z^2)$

In this regime

$$\frac{dV_z}{dt} = -g \quad \text{or} \quad V_z = -gt$$

This is the regime of free fall.

As the velocity increases and approaches V_T now the changes in V_z as a function of time become negligible $\frac{dV_z}{dt} = 0 \Rightarrow V_z = V_T$.

But how does V_z approach V_T?

Consider

$$a_z = -g\left(1 - \frac{V_z^2}{V_T^2}\right)$$

$$= -g\,\frac{V_T^2 - V_z^2}{V_T^2} = -g\,\frac{(V_T - V_z)(V_T + V_z)}{V_T^2}$$

At longer times since $V_z \approx V_T$

we can write $V_T + V \approx 2V_T$

and $V_z - V_T \approx \delta V$

where $\frac{\delta V}{V_T} << 1$

$$a_z = +g \frac{\delta V \; 2V_T}{V_T^{\,2}} = \frac{2g}{V_T} \delta V$$

$$\text{Now} \quad a_z = \frac{dV_z}{dt} = \frac{d}{dt}(\underbrace{V_T}_{\text{const.}} + \delta V) = \frac{d}{dt}\delta V$$

we thus get a simplified differential equation

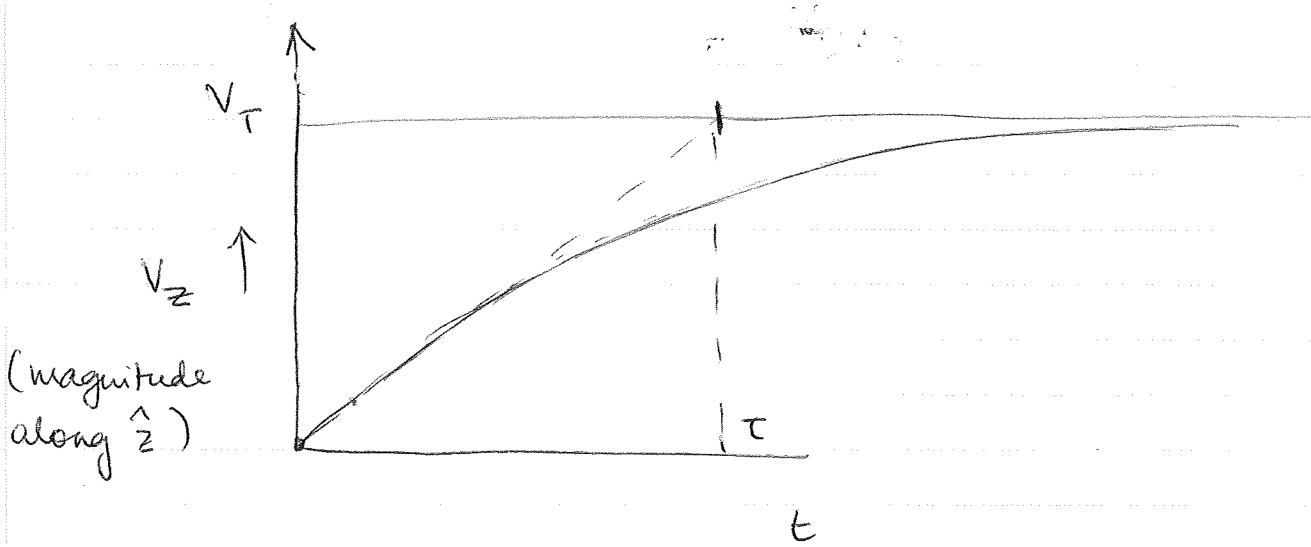
$$\frac{d\delta V}{dt} = - \frac{2g}{V_T} \delta V$$

as $t \uparrow$ δV decreases

$$\Rightarrow \int \frac{d\,\delta V}{\delta V} = -\int \frac{2g}{V_T}\, dt = -\frac{2g}{V_T} t + \text{Const.}$$

$$\Rightarrow \ln \delta V = -\frac{2g}{V_T} t + \text{Const}$$

$$\Rightarrow \delta V = c\exp\left(-\frac{2g}{V_T} t\right)$$



If the motion was free fall

$$V_z = -gt$$

and would equal the terminal velocity after a time $\boxed{\tau = \frac{V_T}{g}}$.

However before that time V_z starts deviating because of the drag and is smaller than expected by drag.

$V_z(t)$ starts approaching V_T exponentially as sketched.

Note the time scale in the exponent works out and τ is the characteristic time.