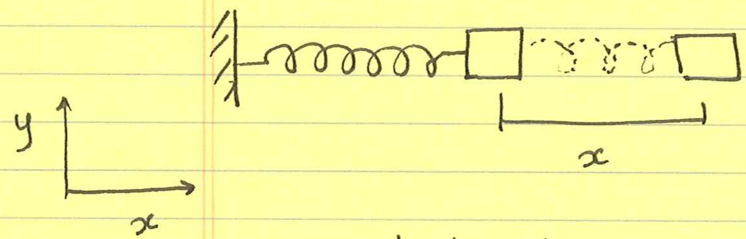
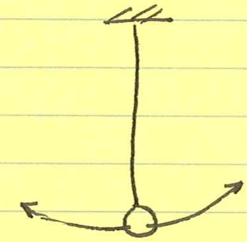
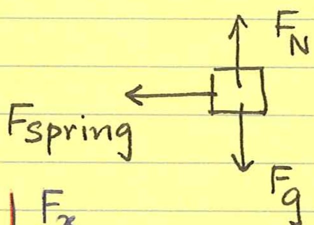
Harmonic Oscillator





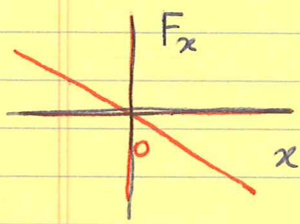
Pendulum

Free body diagram of block



$$\vec{F}_{spring} = -kx\,\hat{x} \qquad (1)$$

Hooke's Law



Equation of motion

$$\vec{F}_{spring} = m\vec{a} \qquad (2)$$

along x $\qquad -kx = m\ddot{x} \qquad (3)$

$$\Rightarrow \boxed{m\ddot{x} + kx = 0} \qquad (4)$$

This is the differential equation that describes the motion. How do we solve it?

We can rewrite (4) as

$$\ddot{x} + \frac{k}{M}x = 0 \qquad (5)$$

Now by dimensional analysis $[\ddot{x}] = \frac{L}{T^2}$

so dimensions of $\frac{K}{M}x = \frac{L}{T^2} \Rightarrow [K/M] = \frac{1}{T^2}$

$\Rightarrow \left[\sqrt{\frac{K}{M}}\right] = \frac{1}{T}$ This quantity is like a frequency

Define $\boxed{\text{NATURAL FREQ. OF OSCILLATOR}}$ $\boxed{\omega_0 = \sqrt{\frac{K}{M}}}$ — (6)

The differential eq. in (5) becomes

$$\boxed{\ddot{x} + \omega_0^2 x = 0} \quad \text{— (7)}$$

Equation of <u>SIMPLE HARMONIC MOTION</u>

Solution is of the form:

$$\boxed{x(t) = B \sin \omega_0 t + C \cos \omega_0 t} \quad \text{— (8)}$$

check:

$$\frac{d}{dt} \sin \omega_0 t = \cos \omega_0 t \, . \, \omega_0 = \omega_0 \cos \omega_0 t$$

$$\frac{d}{dt} \cos \omega_0 t = - \sin \omega_0 t \, . \, \omega_0 = - \omega_0 \sin \omega_0 t$$

$$\dot{x}(t) = B \omega_0 \cos \omega_0 t - C \omega_0 \sin \omega_0 t \quad (9)$$

$$\ddot{x}(t) = - B \omega_0^2 \sin \omega_0 t - C \omega_0^2 \cos \omega_0 t$$

$$= - \omega_0^2 \left[B \sin \omega_0 t + C \cos \omega_0 t \right]$$

$$= - \omega_0^2 \, x(t)$$

which gives Eq. (7) so OK ✓

The constants B & C are evaluated from given initial conditions such as position and velocity at a given time.

Another way to write the solution:

$$x(t) = A \cos(\omega_0 t + \theta) \qquad \text{— (10)}$$

A and θ are constants.

To show the correspondence between (8) and (10) use the trig identity

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$

Apply to (10) which gives:

$$x(t) = A \cos\omega_0 t \cos\theta - A \sin\omega_0 t \sin\theta$$

from (8)

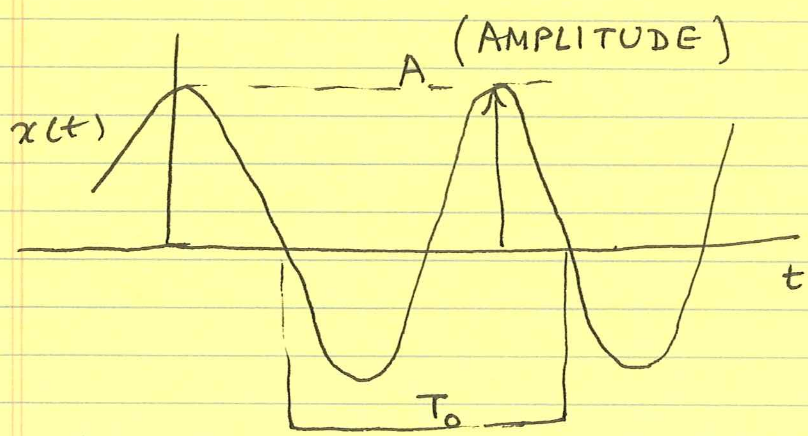
$$= B \sin\omega_0 t + C \cos\omega_0 t$$

Compare the two expressions:

$$B = -A \sin\theta \qquad\qquad A = \sqrt{B^2 + C^2}$$

$$\Rightarrow$$

$$C = A \cos\theta \qquad\qquad \tan\phi = -\frac{B}{C}$$



$$x(t) = A \cos(\omega_0 t + \theta)$$

$$\boxed{\text{Period } T_0 \qquad \omega_0 = \frac{2\pi}{T_0}}$$

$$x(t) = A\cos(\omega_0 t + \theta)$$

instantaneous displacement of particle at time t

amplitude of motion maximum displacement.

ω_0 angular frequency $\qquad \omega_0 = \sqrt{\frac{K}{M}}$

$$[\omega_0] = \frac{\text{rads}}{\text{sec}}$$

$$\text{frequency} = \nu = \frac{\omega_0}{2\pi} \qquad [\nu] = Hz$$

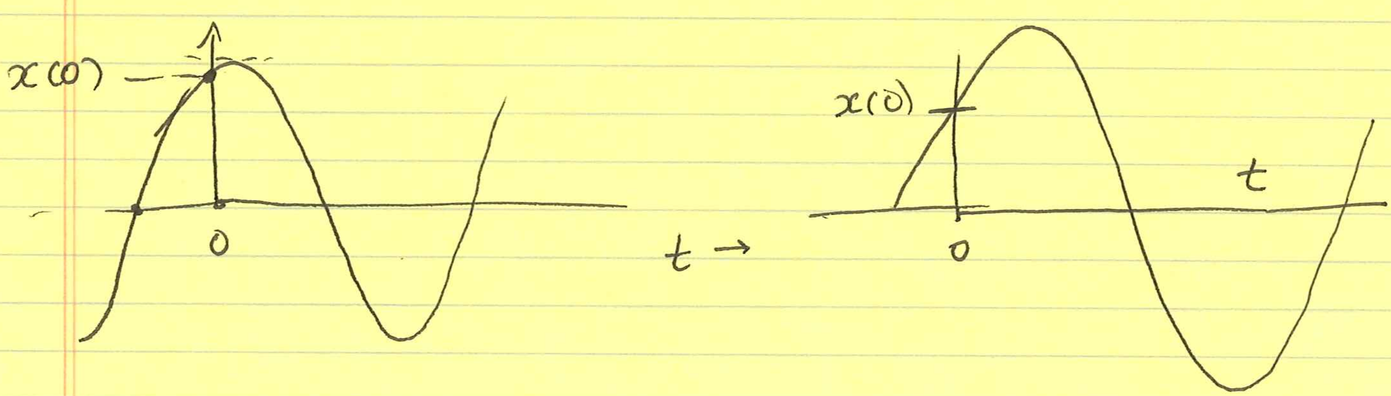
T_0 = period of motion

$$\nu = \frac{1}{T_0} \qquad \omega_0 = \frac{2\pi}{T_0}$$

θ = phase angle

defines the angle at $t = 0$

$$x(0) = A\cos\theta$$



Frictionless Harmonic Oscillator

Initial conditions:

At $t=0$ the position at velocity are given.

$$\left.\begin{aligned} x(t=0) &= x_0 \\ v(t=0) &= \dot{x}(t=0) = v_0 \end{aligned}\right] \text{ given.}$$

$$x = B \sin \omega_0 t + C \cos \omega_0 t$$

$$v = \dot{x} = B\omega_0 \cos \omega_0 t - C\omega_0 \sin \omega_0 t$$

At $t=0$ $\quad x(t=0) = C = x_0$

$$v(t=0) = B\omega_0 = v_0$$

$$\Rightarrow B = \frac{v_0}{\omega_0}$$

If we choose the other form:

$$x = A \cos(\omega_0 t + \theta)$$

$$v = \dot{x} = -A\omega_0 \sin(\omega_0 t + \theta)$$

At $t=0$ $\quad x(t=0) = A\cos\theta = x_0$

$$v(t=0) = -A\omega_0 \sin\theta = v_0$$

$$\Rightarrow A^2(\cos^2\theta + \sin^2\theta) = x_0^2 + \left(\frac{v_0}{\omega_0}\right)^2$$

$$\Rightarrow \boxed{A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_0}\right)^2}} \qquad \boxed{\tan\theta = -\frac{v_0}{\omega_0 x_0}}$$

Energy

$$V = \frac{1}{2} k x^2$$

$$= \frac{1}{2} k A^2 \cos^2(\omega_0 t + \theta)$$

$$\mathbb{K} = \frac{1}{2} m v^2$$

$$= \frac{1}{2} m A^2 \omega_0^2 \sin^2(\omega_0 t + \theta)$$

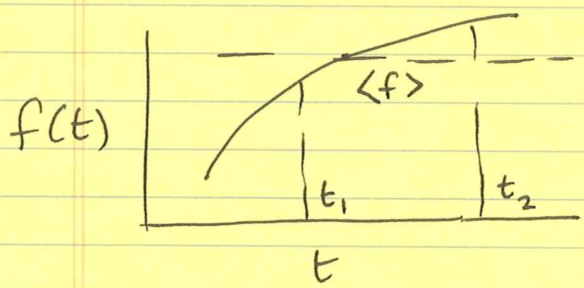
$$\boxed{\omega_0^2 = \frac{k}{m}} \qquad \mathbb{K} = \frac{1}{2} k A^2 \sin^2(\omega_0 t + \theta)$$

Total Energy $E = \mathbb{K} + V = \frac{1}{2} k A^2 \underbrace{\left[\cos^2(\omega_0 t + \theta) + \sin^2(\omega_0 t + \theta)\right]}_{=1}$

$$\boxed{E = \frac{1}{2} k A^2}$$

total energy is constant in time when only conservative forces act.

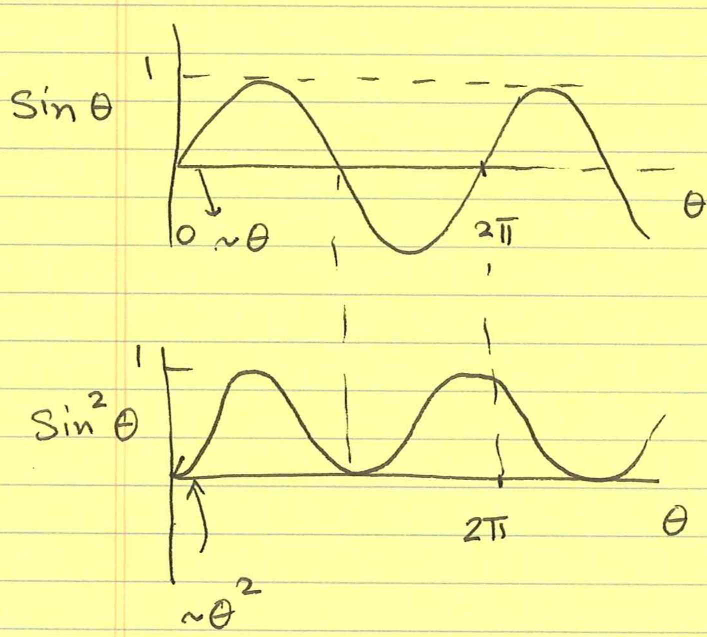
Time average



Consider a function that varies in time.

The average of f denoted by $\langle f \rangle$ over an interval $(t_2 - t_1)$ is given by

$$\boxed{\langle f \rangle = \frac{\int_{t_1}^{t_2} f(t)\, dt}{(t_2 - t_1)}}$$



$\langle \sin\theta \rangle = 0$

$\langle \sin^2\theta \rangle = \frac{1}{2}$

$= \langle \cos^2\theta \rangle$

$$\frac{\int_0^{2\pi} \sin^2\theta \, d\theta}{2\pi} = \langle \sin^2\theta \rangle$$

Now $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= 1 - 2\sin^2\theta$

$\Rightarrow \sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta \left(\frac{1-\cos 2\theta}{2}\right)$$

$$= \frac{1}{2\pi}\left[\frac{1}{2}\, 2\pi - 0\right] = \frac{1}{2} \quad ✓$$

<u>Oscillator:</u>

$$V = \frac{1}{2} k A^2 \cos^2(\omega_0 t + \theta)$$

$$\langle V \rangle = \frac{1}{2} k A^2 \underbrace{\langle \cos^2(\omega_0 t + \theta) \rangle}_{=\frac{1}{2}}$$

average over one time period T_0.

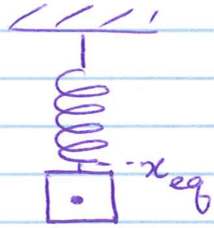
$$= \frac{1}{4} k A^2$$

$$\langle K \rangle = \frac{1}{2} k A^2 \langle \sin^2(\omega_0 t + \theta) \rangle = \frac{1}{4} k A^2$$

$$\Rightarrow \boxed{\langle K \rangle = \langle V \rangle}$$

The time average of kinetic & potential energies is equal.

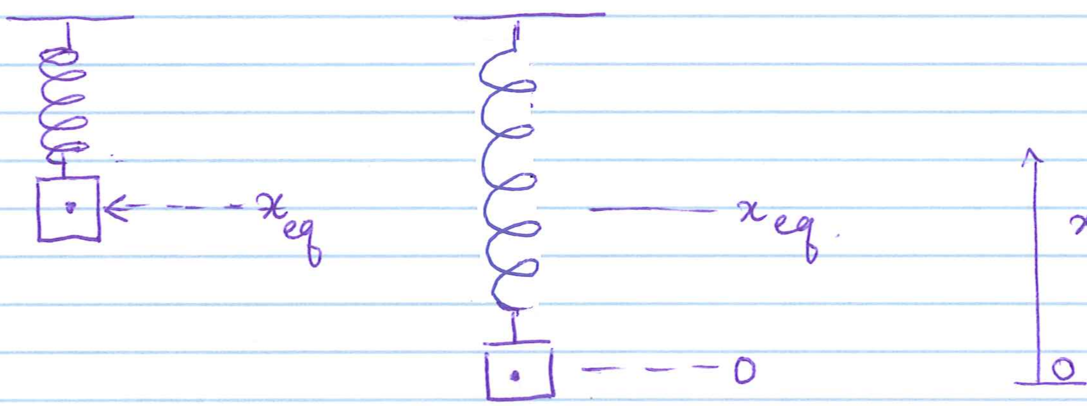
SIMPLE HARMONIC OSCILLATOR



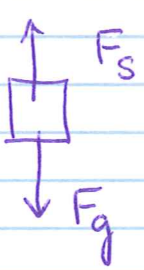
Object hanging on a spring.

x_{eq} is the equilibrium (natural) length of the spring.

When the mass is suspended it will stretch in the gravitational field.



Free body diagram of mass



← note: the force due to the spring on the block is acting up because the spring has been pulled or stretched downwards

At this point there is no acceleration

$$\vec{F_s} + \vec{F_g} = 0$$

$$\vec{F}_s = -k(x - x_{eq})\,\hat{x} \qquad (1)$$

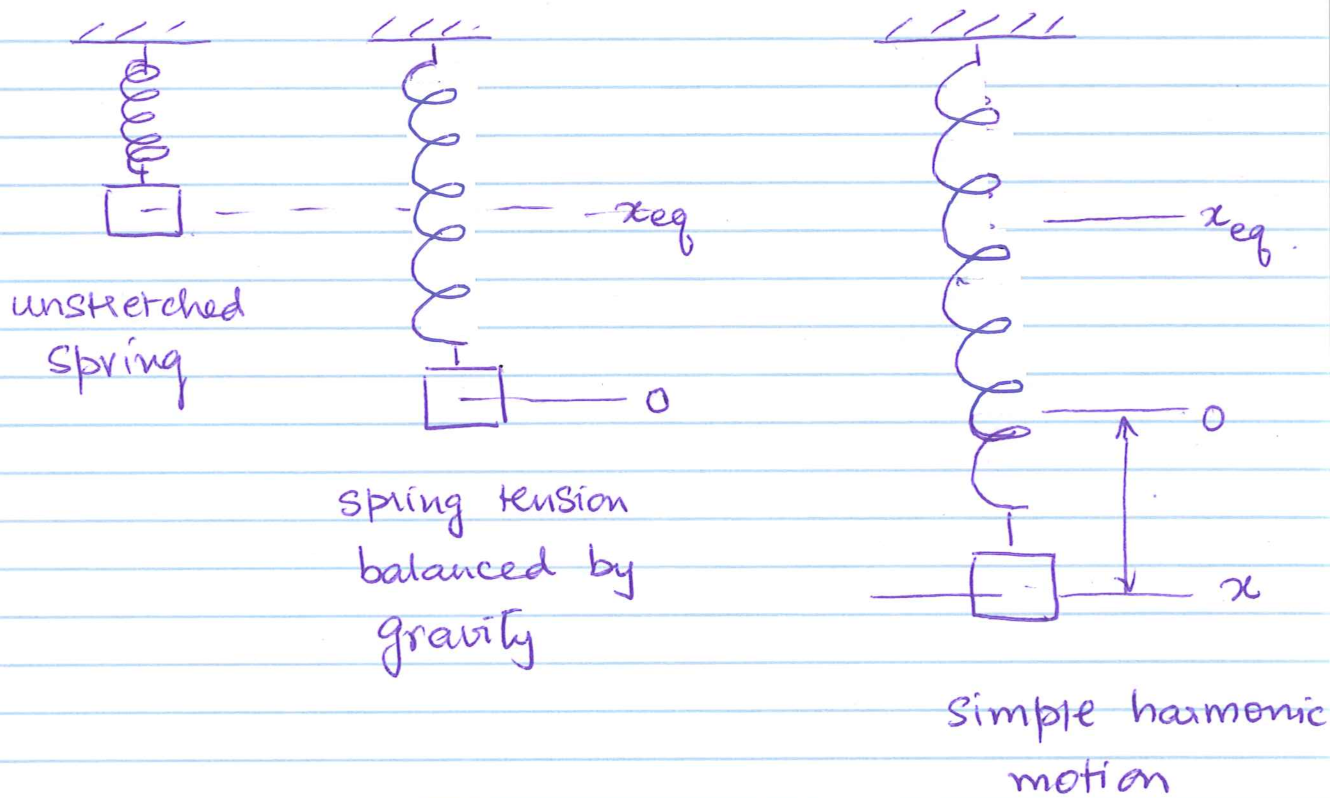
$$\vec{F}_g = -gm\,\hat{x} \qquad (2)$$

$$-k(x - x_{eq}) - mg = 0 \qquad (3)$$

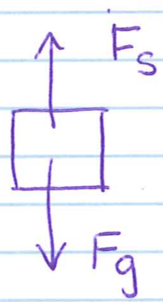
At $x = 0$

$$k x_{eq} = mg \Rightarrow \boxed{x_{eq} = \frac{mg}{k}}$$

Now let us stretch the spring even further



Free body diagram



$$\vec{F}_s + \vec{F}_g = m\vec{a}$$

From (1) and (2) along $\hat{x}$

$$-k(x - x_{eq}) - mg = m\ddot{x}$$

$$-kx + kx_{eq} - mg = m\ddot{x}$$

From (3) $kx_{eq} = mg$

$$\Rightarrow \quad -kx = m\ddot{x}$$

$$\Rightarrow \quad m\ddot{x} + kx = 0$$

or $\boxed{\ddot{x} + \omega_0^2 x = 0}$ $\Rightarrow$ $\boxed{x(t) = A\cos(\omega_0 t + \theta)}$

$\underline{\text{SHM}}$

$$\omega_0^2 = \frac{k}{m}$$