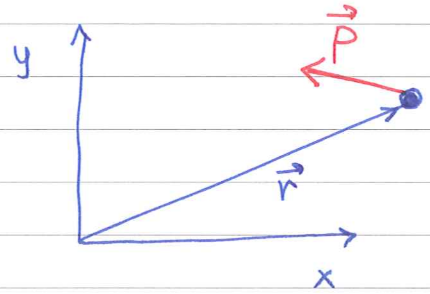
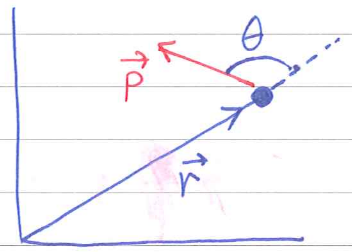
PRECESSION

Angular momentum of a particle $\vec{L} = \vec{r} \times \vec{p}$



To find $\vec{L}$: extend $\vec{r}$ along dotted line

Move the "screw" from $\vec{r}$ toward $\vec{p}$



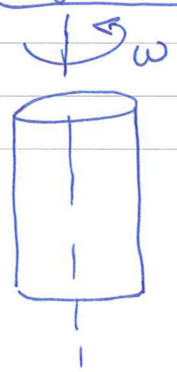
In this case the screw moves out of the paper and defines the direction of $\vec{L}$

Remember : $\vec{L}$ will always be $\perp$ to the plane defined by $\vec{r}$ and $\vec{p}$ (2 vectors define a plane)

To find the magnitude of $\vec{L}$

$$= |\vec{r}|\ |\vec{p}|\ \sin\theta$$

Angular momentum of a rigid object



- define axis of rotation

$$\vec{L} = I\ \vec{\omega}$$

$\vec{\omega}$ ← angular velocity $\vec{\omega} = \frac{d\theta}{dt}\hat{\omega}$

I ↑ moment of inertia around that axis

Torque

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p})$$

Torque equals the rate of change of angular momentum

$$= \frac{d\vec{r}}{dt} \times p + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$= \underbrace{\vec{v} \times \vec{p}}_{=0 \ (\because \vec{v} \parallel \vec{p})} + \vec{r} \times \vec{F}$$

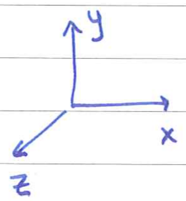
Force equals the rate of change of linear momentum

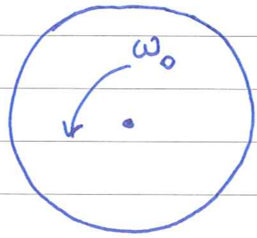
$$\therefore \vec{\tau} = \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}$$

The direction of $\vec{\tau}$ is $\perp$ to the plane defined by $\vec{r}$ and $\vec{F}$. It is also the same as that obtained from the change in angular momentum $d\vec{L}$.

Note: The change $d\vec{L}$ can come from just the magnitude of L changing, or just the direction of L changing keeping the same magnitude, or both changing.

Example 1





Rotating disk of mass M radius R

What is the force required to stop it in time t_0

$$\vec{L}_{initial} = I\,\vec{\omega}_{initial}$$

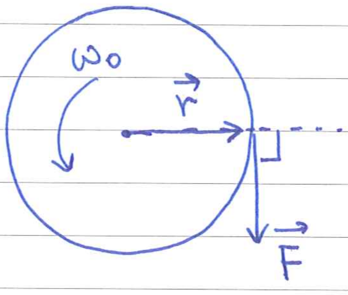
$$= \frac{1}{2} M R^2 \omega_0 \hat{z}$$

$$\vec{L}_{final} = 0$$

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \frac{L_{final} - L_{initial}}{dt} = \frac{-\frac{1}{2} M R^2 \omega_0 \hat{z}}{t_0} \quad \text{— (1)}$$

Now to get the force

$$\vec{\tau} = \vec{r} \times \vec{F} = R\,F\,\sin 90^\circ\,(-\hat{z}) = -RF\,\hat{z} \quad \text{— (2)}$$



Equating (1) & (2)

$$\frac{\frac{1}{2} M R^2 \omega_0}{t_0} = R\,F$$

$$\Rightarrow \boxed{F = \frac{M R \omega_0}{2 t_0}}$$

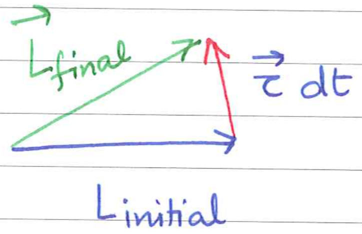
check units: $[F] = \left[\frac{ML}{T^2}\right] = [N]$ ✓

PRECESSION OF A TOP

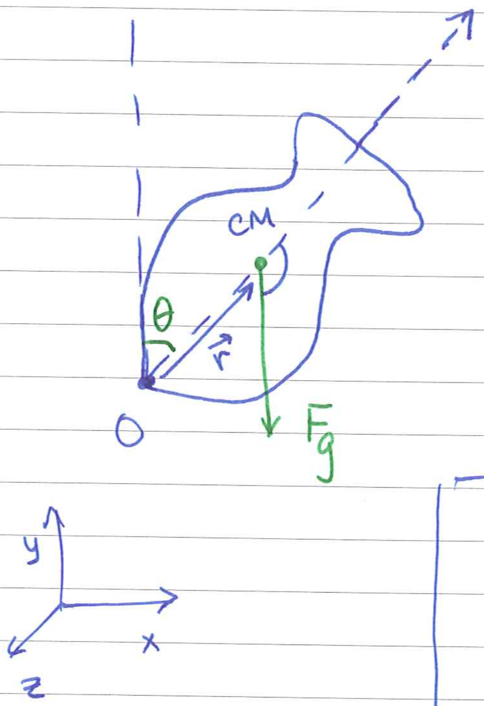
classic situation when $\vec{\tau}$ is not parallel to $\vec{L}$

$$\vec{\tau} = \frac{d\vec{L}}{dt} \Rightarrow d\vec{L} = \vec{\tau}\, dt$$

$$\Rightarrow \vec{L}_{final} = \vec{L}_{initial} + \vec{\tau}\, dt$$



Consider a top on a table at an angle θ



$F_g = mg$ exerts a torque at O

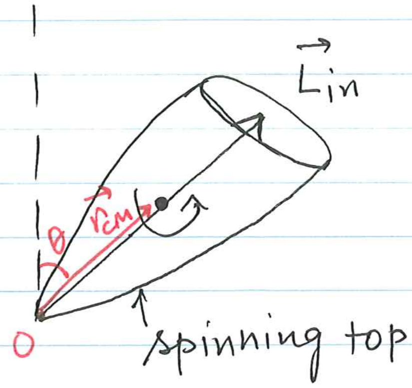
$$\vec{\tau} = \vec{r}_{CM} \times \vec{F}_g$$

$$= r_{CM} F_g \sin(\pi - \theta)\,(-\hat{z})$$

$$\vec{\tau} = r_{CM} F_g \sin\theta\,(-\hat{z})$$

UNDER THE ACTION OF THIS TORQUE IN THE CLOCKWISE DIRECTION A STATIONARY TOP WILL TOPPLE OVER

What happens when you release a spinning top?



The top has an initial angular momentum $\vec{L}_{in}$

The direction of $\vec{L}_{in} \parallel \vec{r}_{CM}$

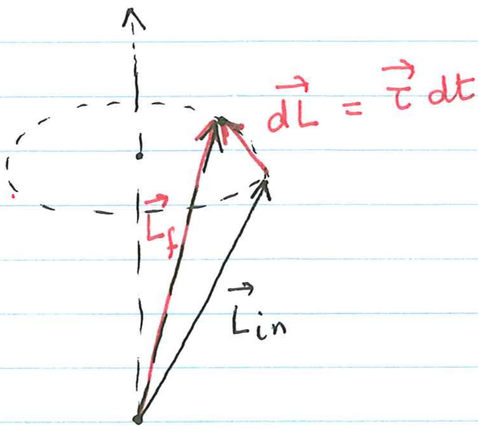
Since $\vec{\tau} = \vec{r}_{CM} \times \vec{F}$

$\Rightarrow \vec{\tau} \perp$ to both $\vec{r}_{CM}$ & $\vec{F}$

$\Rightarrow \vec{\tau} \perp \vec{L}_{in}$

The torque due to gravity is $\perp$ to $\vec{L}_{in}$

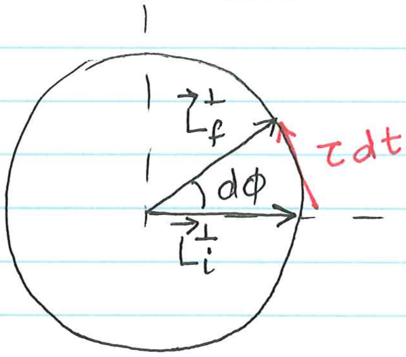
$\Rightarrow$ the torque cannot change the magnitude of $\vec{L}_{in}$ but only its direction



With the same opening angle θ remaining unchanged the tip charts out a cone

This is called precession

Top view:



$\vec{L}_i^\perp$ is the projection of $\vec{L}_i$ on the circle. Its length equals $L_i \sin\theta$

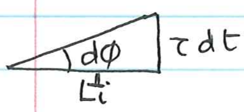
In a time dt $\vec{L}_i^\perp$ moves to $\vec{L}_f^\perp$. The angle

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

$$d\phi = \frac{dL}{L_i^\perp} = \frac{\tau\, dt}{L_i^\perp}$$

$$= \frac{r_{CM} F_g \overbrace{\sin(\pi - \theta)}^{= \sin\theta} dt}{L \sin\theta}$$

To understand this



$$\sin\theta \simeq \tan\theta \simeq \theta = \frac{\tau dt}{L_i^\perp}$$

Small angle formula.

$$\boxed{\frac{d\phi}{dt} = \frac{r_{CM} F_g}{I\omega}}$$

precession rate ($\frac{d\phi}{dt}$)

angular velocity (ω)

Note $\underline{\underline{\sin\theta}}$ cancels out so the precession rate does not depend on the angle of tilt of the top.