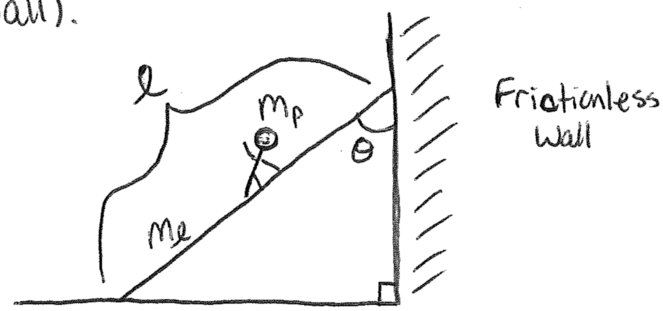
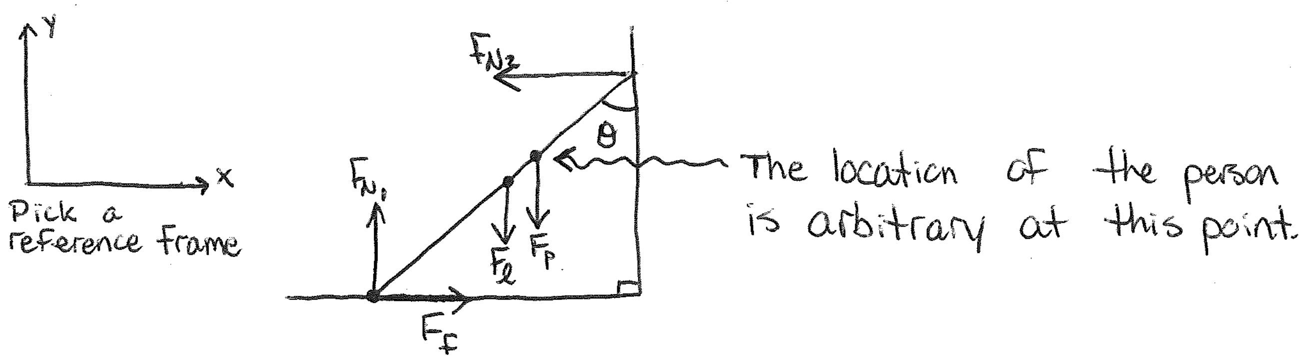
Ladder w/ a person Problem

If a person (mass (m_p)) climbs on a ladder of given mass (m_ℓ), length (ℓ), angle (θ) with respect to the wall, and coefficient of friction (μ), what is the highest the person can climb without the ladder slipping? (Assume frictionless wall).

These are the givens:



Next, find and label the forces acting on this system.



The net sum of the forces' components will equal zero.

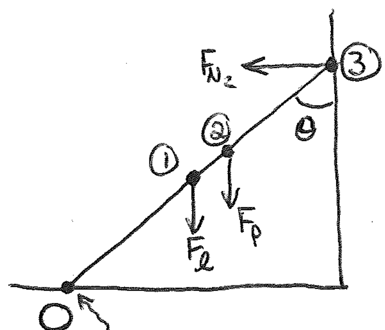
$$\vec{F}_{N_1} + \vec{F}_f + \vec{F}_\ell + \vec{F}_p + \vec{F}_{N_2} = 0$$

$$\begin{pmatrix} 0 \\ F_{N_1} \end{pmatrix} + \begin{pmatrix} F_f \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -F_\ell \end{pmatrix} + \begin{pmatrix} 0 \\ -F_p \end{pmatrix} + \begin{pmatrix} -F_{N_2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solve and set the forces equal to each other.

$$F_f = F_{N_2} \quad \text{and} \quad F_{N_1} = F_\ell + F_p$$

Now, we want to set the torques equal so that they balance. First, find how many torques are acting on the system.

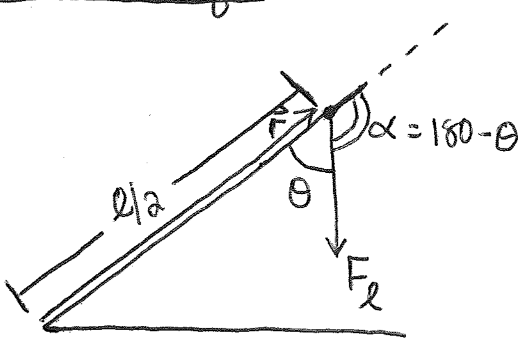


There are $\underline{\underline{3}}$ torques acting on the system.

I will pick this point as my origin ("gets rid of" two of the forces)

We want to find each individual torque, so draw a separate picture for each torque.

<u>Ladder Torque</u>



$$\tau_\ell = \vec{r} \times \vec{F}_\ell = rF_\ell \sin\alpha\,(-\hat{z})$$ ← screw goes into board

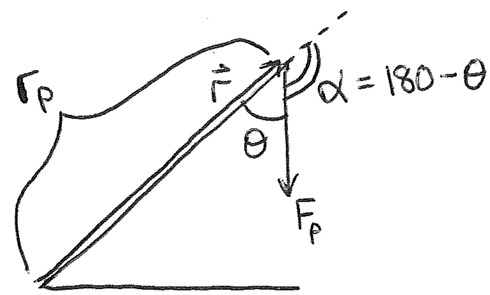
$$= \left(\frac{\ell}{2}\right)(m_\ell g)\sin(180-\theta)(-\hat{z})$$

$$= \frac{\ell}{2} \cdot m_\ell g \sin\theta\,(-\hat{z})$$

Useful Trig.

$$\sin(90+\theta) = \cos\theta \quad \text{and} \quad \sin(180-\theta) = \sin\theta$$

Person Torque



$$\tau_p = \vec{r} \times \vec{F}_p$$

$$= r F_p \sin\alpha \, (-\hat{z})$$ ← screw goes into board

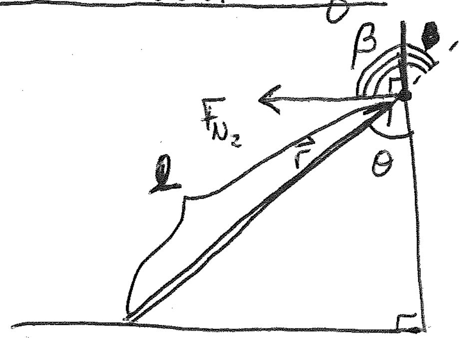
$$= r_p (m_p g) \sin(180 - \theta)(-\hat{z})$$

$$= r_p (m_p g) \sin\theta \, (-\hat{z})$$

SPOILER ALERT:

r_p is what we are solving for ☺

Wall-ladder Torque



$\beta = 90 + \theta$

$$\tau_w = \vec{r} \times \vec{F}_{N_2} = r F_{N_2} \sin\beta \, (+\hat{z})$$ ← screw comes out of board

$$= (\ell)(F_{N_2}) \sin(90 + \theta)(+\hat{z})$$

$$= \ell F_{N_2} \cos\theta \, (+\hat{z})$$

What is F_{N_2} ???

... Glad you asked...

Remember...

$$F_{N_2} = F_F$$

And we know that

$$F_{F,max} = \mu F_{N_1}$$

So, with that...

$$F_{N_2} = F_F = \mu F_{N_1} = \mu(m_\ell g + m_p g) = \mu g \underbrace{(m_\ell + m_p)}_{M} = \mu g M$$

Back to the wall-ladder torque...

$$\tau_w = \ell F_{N_2} \cos\theta \, (+\hat{z})$$
$$= \ell(\mu g M)\cos\theta \, (+\hat{z})$$

Time to set your torques equal. Set the $(+z)$ equal to the $(-\hat{z})$ side and you get...

$$\ell \mu g M \cos\theta = \frac{\ell}{2} m_\ell g \sin\theta + r_p m_p g \sin\theta$$

Remember the spoiler alert? Yeah, now it's just algebra.

$$\ell \mu \cancel{g} M \cos\theta = \frac{\ell}{2} m_\ell \cancel{g} \sin\theta + r_p m_p \cancel{g} \sin\theta$$

$$\ell \mu M \cos\theta = \sin\theta \left(\frac{\ell}{2} m_\ell + r_p m_p\right)$$

$$\ell \mu M \cot\theta = \frac{\ell}{2} m_\ell + r_p m_p$$

$$\ell \mu M \cot\theta - \frac{\ell}{2} m_\ell = r_p m_p$$

And Finally...

$$\boxed{\Gamma_p = \frac{\ell \mu M \cot\theta - \frac{\ell}{2} m_\ell}{m_p}}$$

We all know how to plug n' chug from here...

... But wait. How do we know this works?

Let's look at the units and analyze a few things.

$$\Gamma_p = \frac{\ell \mu M \cot\theta - \frac{\ell}{2} m_\ell}{m_p}$$

$$\Gamma_p = \ell \left[\mu \left(\frac{M}{m_p} \right) \cot\theta - \frac{1}{2} \left(\frac{m_\ell}{m_p} \right) \right]$$

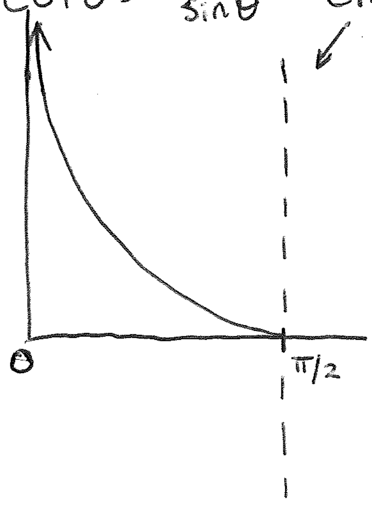
$$[m] = [m] \left[[\ddot{\frown}] \left[\frac{\cancel{kg}}{\cancel{kg}} \right] [\ddot{\frown}] - [\ddot{\frown}] \left[\frac{\cancel{kg}}{\cancel{kg}} \right] \right] \quad ; [\ddot{\frown}] = \text{unitless}$$

$$[m] = [m] \ \checkmark$$

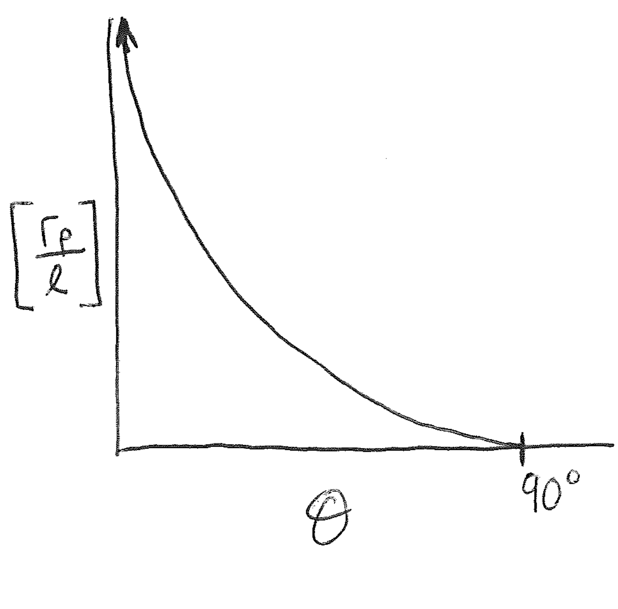
What does cotangent have to do with this?

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Graph



And let's say the ladders mass is negligable



This graph tells us that as θ becomes smaller the person will be able to climb higher on the ladder because the torques are not acting as strong on the system. Vis-a-vis, the larger θ is, the less high the person can climb up the ladder.