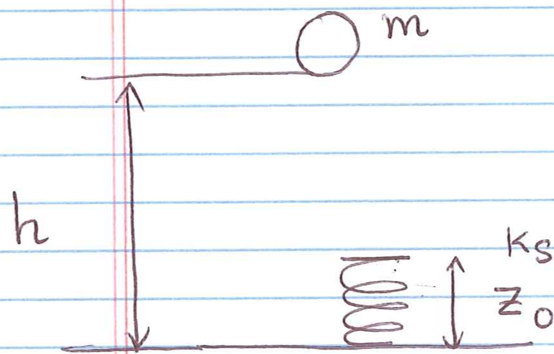
# Spring Problems

Consider the following problem:

A ball (and just to keep it simple, assume it is not rotating) falls from a height $h$ on a coiled spring of spring constant $k_s$. How much does the spring compress?

$m, k_s, z_0, h$ given




$\boxed{\text{Find } z}$

$\boxed{\text{Find } v_f \text{ when ball just hits the spring.}}$

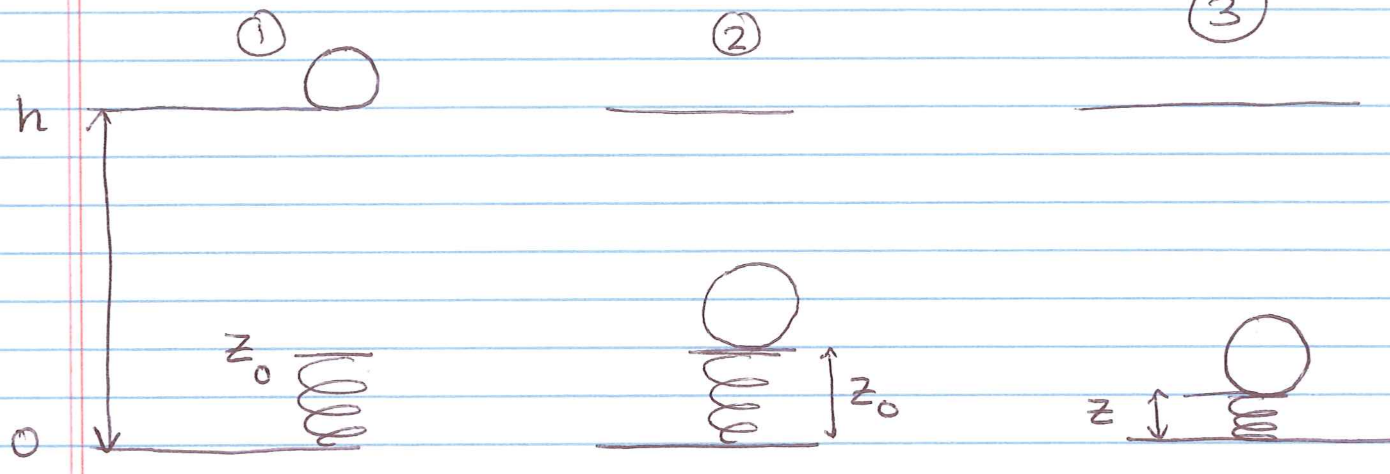
There are two ways to set it up and you can decide which one you prefer.

In the first, I pick the reference for the potential energy due to gravity on the ground.

In the second, I pick the reference at the position of the unstretched spring.

Assume the ball is a <u>point particle</u>

Reference I




ball at top | ball just before it hits spring | ball after full compression of spring.

gravity

$$E_1 = \underset{0}{\cancel{K_1}} + V_1 + \underset{0}{\cancel{V_{spring,1}}} \qquad E_2 = K_2 + V_2 + \underset{0}{\cancel{V_{s,2}}}$$

$$= \frac{1}{2} m v_f^2 + mgz_0$$

$$= mgh$$

$$E_1 = E_2 \Rightarrow mgh = \frac{1}{2} m v_f^2 + mgz_0$$

$$\Rightarrow \boxed{v_f^2 = 2g(h - z_0)}$$

$$\text{or} \quad v_f = \sqrt{2g(h - z_0)}$$

$$E_3 = \underbrace{\cancel{K_3}}_{0} + \overset{\text{gravity}}{V_3} + \overset{\text{Spring}}{V_{s,3}}$$

$$= mgz + \frac{1}{2} k_s (z - z_0)^2$$

We can ~~eig~~ either equate $E_2 = E_3$ or $E_1 = E_3$ to calculate $z$. I will choose $E_1 = E_3$ as that is easier.

$$mgh = mgz + \frac{1}{2} k_s (z - z_0)^2$$

$$\Rightarrow \quad \frac{1}{2} k_s (z - z_0)^2 + mgz - mgh = 0$$

This is a quadratic equation that I have to solve to get $(z - z_0)$. I need to fix the variable though in the second term.

$$\frac{1}{2} k_s (z - z_0)^2 + mg(z - z_0) + mg z_0 - mgh = 0$$

$$\boxed{\frac{1}{2} k_s (z - z_0)^2 + mg(z - z_0) \cancel{+} - mg(h - z_0) = 0}$$

This is of the form $ax^2 + bx + c = 0$

whose solution is $x = \dfrac{-a \pm \sqrt{b^2 - 4ac}}{2a}$

with $x = z - z_0$ $\quad a = k_s/2 \quad b = mg$

$c = -mg(h - z_0)$

$$z - z_0 = \frac{-mg \pm \sqrt{(mg)^2 + 4(\frac{1}{2}k_s)mg(h-z_0)}}{2(\frac{1}{2}k_s)}$$

must pick negative sign because $z$ is less than $z_0$.

$$= \frac{-mg \pm mg\left(1 + \frac{2k_s(h-z_0)}{mg}\right)^{1/2}}{k_s}$$

c <u>check units.</u>

We have $[mg\cancel{\ell}] = [k_s \cancel{\ell}^2]$

$$\Rightarrow [\text{length}] = \left[\frac{mg\cancel{\ell}}{k_s}\right]$$

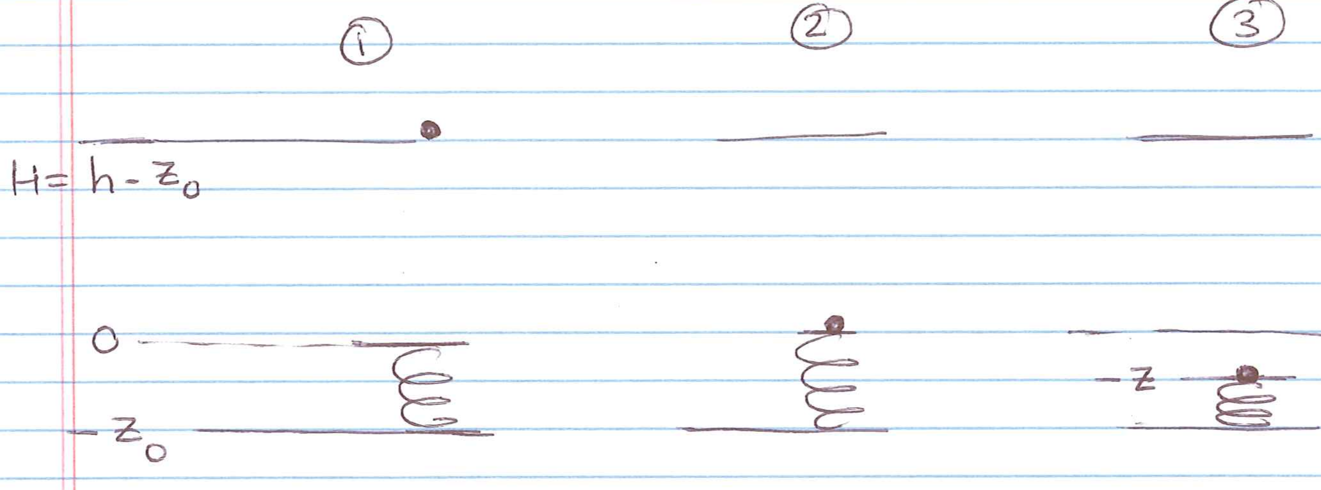
i.e. $\frac{mg}{k_s}$ has units of length.

So also in the term $\frac{k_s(h-z_0)}{mg}$ should ~~there~~ be dimensionless and that works out.

We can mold the expression above a little more:

$$\boxed{z - z_0 = -\frac{mg}{k_s}\left[1 + \left\{1 + \frac{2k_s}{mg}(h-z_0)\right\}^{1/2}\right]}$$

## Reference II




$$E_1 = V_1 + \cancelto{0}{K_1} + \cancelto{0}{V_{s,1}}$$

$$= mgH$$

$$E_2 = K_2 + V_2 + \cancelto{0}{V_{s,2}}$$

$$= \frac{1}{2} m v_f^2 + \cancelto{0}{mg(0)}$$

$E_1 = E_2$

$$\Rightarrow \quad mgH = \frac{1}{2} m v_f^2$$

$$\Rightarrow \quad v_f^2 = 2gH$$

(which is the same result as before)

$$E_3 = E_1$$

$$\Rightarrow \quad \cancelto{0}{K_3} + V_3 + V_{s,3} = mgH$$

$$\frac{1}{2} k_s z^2 - mgz = mgH$$

$$\Rightarrow \quad \boxed{\frac{1}{2} k_s z^2 - mgz - mgH = 0}$$

$$z = \frac{mg \pm \sqrt{(mg)^2 + 4\frac{k_s}{2} mg H}}{2\left(\frac{k_s}{2}\right)}$$

$$z = \frac{mg \pm mg\left(1 + \frac{2 k_s H}{mg}\right)^{1/2}}{k_s}$$

$$= \frac{mg}{k_s}\left[1 \pm \left(1 + \frac{2 k_s H}{mg}\right)^{1/2}\right]$$

~~While~~ we expect the compression to be negative, but we already incorporated the sign in our expressions e.g. when we wrote $mg(-z)$ for the potential energy. So in the quadratic equation ~~gives the~~ $z > 0$.

$$z = \frac{mg}{k_s}\left[1 + \left(1 + \frac{2 k_s H}{mg}\right)^{1/2}\right]$$

Which is also equal to the expression by reference I provide we make the appropriate substitutions $H = h - z_0$