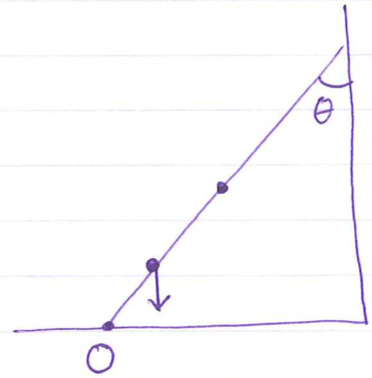
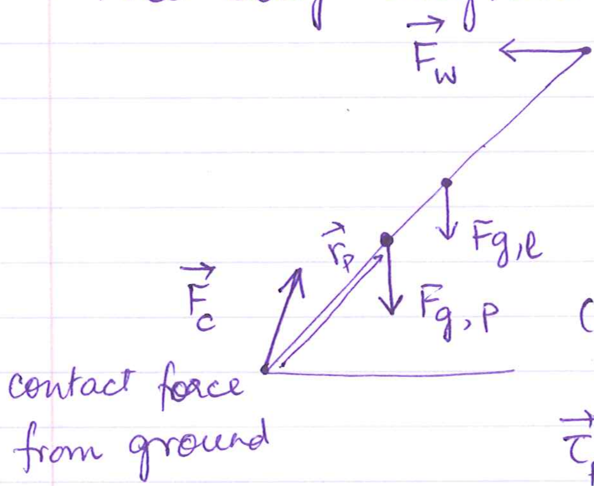
N5B.4



Free Body diagram of ladder

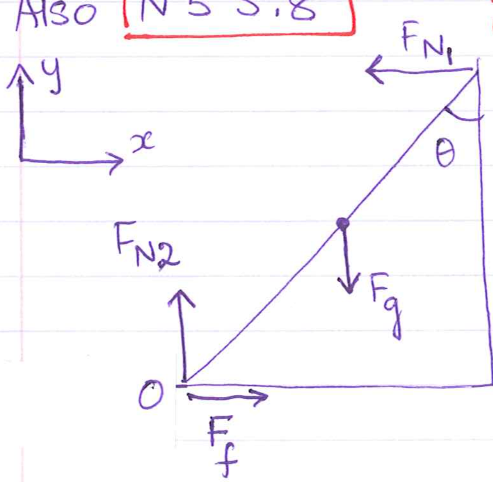


$F_{g,l}$ (gravity, due to ladder)

$F_{g,p}$ (gravity due to person)

$$\vec{\tau}_p = \vec{r}_p \times \vec{F}_g \quad \text{(clockwise)}$$

Also $\boxed{\text{N5S.8}}$ ← we are given frictionless surface with wall



$\vec{F}_{N_1}$ = normal force from wall

$\vec{F}_{N2}$ = normal force from ground

$$\vec{F}_{N_1} + \vec{F}_{N_2} + \vec{F}_g + \vec{F}_f = 0$$

$$\begin{pmatrix} -F_{N_1} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ F_{N_2} \end{pmatrix} + \begin{pmatrix} 0 \\ -F_g \end{pmatrix} + \begin{pmatrix} F_f \\ 0 \end{pmatrix}$$

$$= 0$$

$$F_{N_1} = F_f$$

$$F_{N_2} = F_g$$

Now use torque equations: Pick the origin about O. (ground)

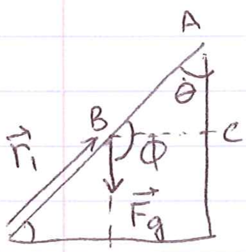
$$\underbrace{\vec{r}_1 \times \vec{F}_g}_{\text{clockwise}} + \underbrace{\vec{r}_2 \times \vec{F}_{N_1}}_{\text{anticlockwise}} = 0.$$

($\vec{F}_f$ and $\vec{F}_{N_1}$ don't exert any torque because the lever arm is zero)

$$|\vec{r}_1| = L/2$$

$$|\vec{r}_2| = L$$

$$|\vec{r}_1 \times \vec{F}_g| = \frac{L}{2} F_g \sin\phi = \frac{L}{2} F_g \sin(180-\theta) = \frac{L}{2} F_g \sin\theta.$$



How is ϕ related to θ?

given

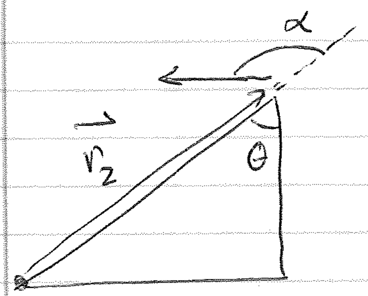
$$\angle ABC = 90^\circ - \theta$$

$$\phi = 90 + \angle ABC$$

$$= 90 + 90 - \theta$$

$$= 180 - \theta$$

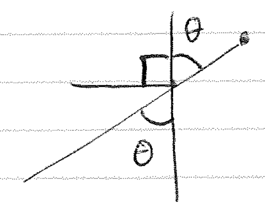
$$\left|\vec{r}_2 \times \vec{F}_{N_1}\right| = r_2 F_{N_1} \sin\alpha$$



$$= r_2 F_{N_1} \sin(90+\theta)$$

$$= r_2 F_{N_1} \cos\theta = L F_{N_1} \cos\theta$$

How is α related to θ ?



$$\alpha = 90 + \theta$$

Now putting the two torques together

$$\frac{L}{2} F_g \sin\theta = L F_{N_1} \cos\theta$$

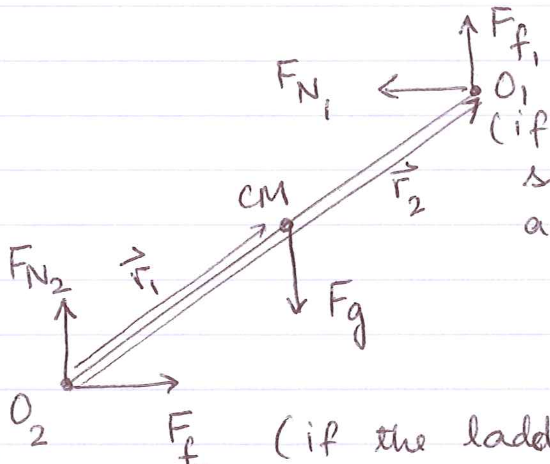
$$\Rightarrow \boxed{F_{N_1} = \frac{1}{2} F_g \tan\theta} = F_f$$

$$\boxed{F_{N_2} = F_g}$$

In-class discussion

We have included friction at the contact point of the ladder with the floor — what about friction at the contact point of the ladder with the wall? [in NS 5.8 we were given that there is no friction with wall but what if friction is not zero]

Let's add that in our analysis below —



(if the ladder were to slide down, friction would act in the up direction)

(if the ladder were to slide the bottom of the ladder would move left, hence friction acts to the right)

We have to determine F_{N_1}, F_{N_2}, F_{f_1}, F_{f_2}

$$\begin{pmatrix} -F_{N_1} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ F_{N_2} \end{pmatrix} + \begin{pmatrix} 0 \\ -F_g \end{pmatrix} + \begin{pmatrix} F_{f_2} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ F_{f_1} \end{pmatrix} = 0$$

$$\Rightarrow \boxed{\begin{aligned} F_{f_2} &= F_{N_1} && \text{—— (1)} \\ F_{f_1} + F_{N_2} &= F_g && \text{—— (2)} \end{aligned}}$$

Next use torque equations to find more equations to solve for the unknown forces.

We can pick any origin. Use ful to pick either O_1 or O_2 or both (because that eliminates the torques from the two forces F_N and F_f that pass through the origin).

If we pick the origin to be CM of the ladder then F_g which passes through CM will not contribute to the torque but all the 4 unknown forces F_{N_1}, F_{N_2}, F_{f_1}, F_{f_2} will contribute.

Pick origin about O_2 (ground)

$$\vec{r}_1 \times \vec{F}_g + \vec{r}_2 \times \vec{F}_{N_1} + \vec{r}_2 \times \vec{F}_{f_1} = 0$$

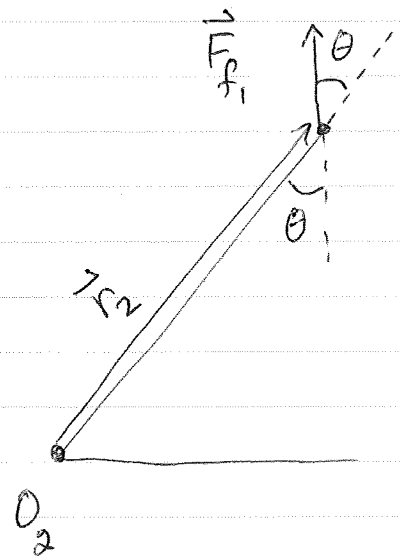
$$\vec{r}_1 \times \vec{F}_g = \frac{L}{2} F_g \sin\theta \quad \text{clockwise}$$

(evaluated on page 12)

$$\vec{r}_2 \times \vec{F}_{N_1} = L F_{N_1} \cos\theta \quad \text{anticlockwise}$$

(evaluated on page 13)

Torque due to $\vec{F}_{f_1}$



$$\vec{r}_2 \times \vec{F}_{f_1} = L\, F_{f_1} \sin\theta$$

(anticlockwise)

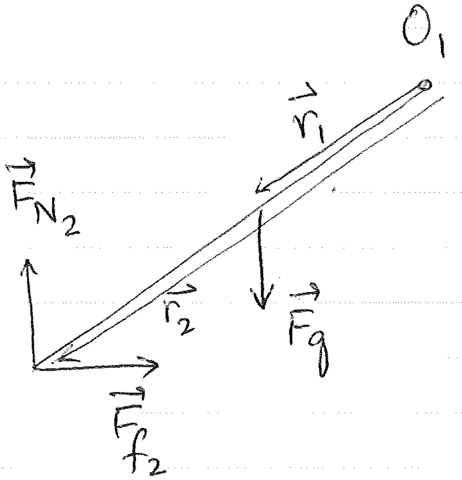
$$\Rightarrow \quad L\, F_{N_1} \cos\theta + L\, F_{f_1} \sin\theta = \frac{L}{2} F_g \sin\theta$$

$$\Rightarrow \quad \boxed{F_{N_1} \cos\theta + F_{f_1} \sin\theta = \frac{F_g}{2} \sin\theta} \quad -(3)$$

We now have (3) equations and 4 unknowns (F_{N_1}, F_{N_2}, F_{f_1}, F_{f_2})

So we need one more equation.

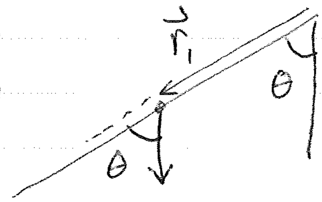
We check out torques about O_1 and that will give us the 4th equation.



Torques about O_1

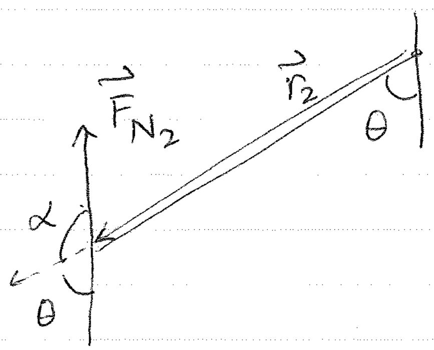
$$\vec{r}_1 \times \vec{F}_g + \vec{r}_2 \times \vec{F}_{N_2} + \vec{r}_2 \times \vec{F}_{f_2} = 0$$

$\vec{r}_1 \times \vec{F}_g$



$$\vec{r}_1 \times \vec{F}_g = \frac{L}{2} F_g \sin\theta$$

(anticlockwise)



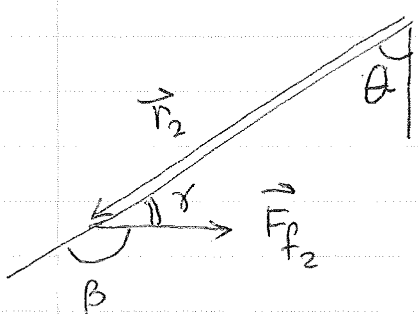
$$\vec{r}_2 \times \vec{F}_{N_2} = L F_{N_2} \sin\alpha$$

(clockwise)

$$= L F_{N_2} \sin(180-\theta)$$

$$= L F_{N_2} \sin\theta.$$

$\alpha = 180 - \theta$



$$\vec{r}_2 \times \vec{F}_{f_2} = L F_{f_2} \sin\beta$$

(anticlockwise)

$$= L F_{f_2} \sin(90+\theta) = L F_{f_2} \cos\theta$$

$\gamma = 90 - \theta$

$\beta = 180 - \gamma = 180 - (90 - \theta)$
$= 90 + \theta$

Putting all the torques together

$$\frac{L}{2} F_g \sin\theta - L F_{N_2} \sin\theta + L F_{f_2} \cos\theta = 0$$

$$\Rightarrow \boxed{\frac{1}{2} F_g \sin\theta - F_{N_2} \sin\theta + F_{f_2} \cos\theta = 0}$$

(4.)

Let's write all 4 equations:

$$F_{f_2} = F_{N_1} \quad \text{--- (1)}$$

$$F_{f_1} + F_{N_2} = F_g \quad \text{--- (2)}$$

$$F_{N_1} \cos\theta + F_{f_1} \sin\theta = \frac{F_g}{2} \sin\theta \quad \text{--- (3)}$$

$$\frac{F_g}{2} \sin\theta - F_{N_2} \sin\theta + F_{f_2} \cos\theta = 0 \quad \text{--- (4)}$$

In Eq. (3) we can eliminate F_{N_1} using (1) and get

$$F_{f_2} \cos\theta + F_{f_1} \sin\theta = \frac{F_g}{2} \sin\theta$$

$$\Rightarrow \boxed{F_{f_2} + F_{f_1} \tan\theta = \frac{F_g}{2} \tan\theta} \qquad (5)$$

which is an equation that connects the two frictional forces in terms of F_g & θ.

From Eq. (4) also if we eliminate F_{N_2} using (2) we get

$$\frac{F_g}{2} \sin\theta - \left(F_g - F_{f_1}\right) \sin\theta + F_{f_2} \cos\theta = 0$$

$$\Rightarrow \frac{F_g}{2} \sin\theta - F_g \sin\theta + F_{f_1} \sin\theta + F_{f_2} \cos\theta = 0$$

$$\Rightarrow -\frac{F_g}{2} \sin\theta + F_{f_1} \sin\theta + F_{f_2} \cos\theta = 0$$

$$\Rightarrow F_{f_2} + F_{f_1} \tan\theta = \frac{F_g}{2} \tan\theta$$

which is the same as Eq. (5).

Thus taking torques about O_1 does not give an independent equation that is different from that obtained by taking torques about O_2.
You can also show that taking torques about the CM also does not yield an independent equation.

So finally we have:

$$F_{N_1} = F_{f_2} \quad (1)$$

$$F_{N_2} = F_g - F_{f_1} \quad (2)$$

normal forces in terms of frictional forces

and $$F_{f_1} \tan\theta + F_{f_2} = \frac{F_g}{2} \tan\theta \quad (5)$$

an equation that relates the two frictional forces.

we can make more progress if we next use the fact that the maximum static friction $F_{sf}\big|_{max} = \mu_s N$

$$F_{f_1} = \mu_{s_1} F_{N_1} \quad (6)$$

↑ coefficient of static friction between ladder and wall

$$F_{f_2} = \mu_{s_2} F_{N_2} \quad (7)$$

↑ coefficient of static friction between ladder and floor.

$$F_{N_1} = \mu_2 F_{N_2} \qquad \text{using (1) \& (7)}$$

$$= \mu_2 (F_g - \mu_1 F_{N_1}) \qquad \text{using (2) \& (6)}$$

$$= \mu_2 F_g - \mu_1 \mu_2 F_{N_1}$$

$$\Rightarrow \quad F_{N_1} (1 + \mu_1 \mu_2) = \mu_2 F_g$$

$$\Rightarrow \quad \boxed{F_{N_1} = \frac{\mu_2}{1 + \mu_1 \mu_2} F_g} \quad \text{— (8)}$$

$$\boxed{F_{N_2} = \frac{F_{N_1}}{\mu_2} = \frac{1}{1 + \mu_1 \mu_2} F_g} \quad \text{— (9)}$$

Eqns (8) and (9) were obtained from the force balance equations along x and y and the constitutive friction eqns (6) & (7).

Let us now see the constraint that Eq. (5) that relates the two frictional forces produces.

$$F_{f_1} \tan\theta + F_{f_2} = \frac{F_g}{2} \tan\theta$$

$$\mu_1 F_{N_1} \tan\theta + \mu_2 F_{N_2} = \frac{F_g}{2} \tan\theta$$

Substitute for F_{N_1} from (8) and F_{N_2} from (9)

$$\frac{\mu_1 \mu_2}{1 + \mu_1 \mu_2} F_g \tan\theta + \frac{\mu_2}{1 + \mu_1 \mu_2} F_g = \frac{F_g}{2} \tan\theta$$

$$\left(\frac{\mu_1 \mu_2}{1 + \mu_1 \mu_2} - \frac{1}{2} \right) \tan\theta + \frac{\mu_2}{1 + \mu_1 \mu_2} = 0$$

$$\Rightarrow \frac{2\mu_1 \mu_2 - (1 + \mu_1 \mu_2)}{2(1 + \mu_1 \mu_2)} \tan\theta + \frac{\mu_2}{1 + \mu_1 \mu_2} = 0$$

$$\Rightarrow \frac{\mu_1 \mu_2 - 1}{2} \tan\theta = -\mu_2$$

$$\Rightarrow \boxed{\tan\theta = \frac{2\mu_2}{1 - \mu_1 \mu_2}} \quad - (10)$$

Final solutions are contained in Eqs. (8), (9), (10)

(a) Assume frictionless surface between ladder and wall $\underline{\mu_1 = 0}$

Eq. (8) $\Rightarrow$ $$F_{N_1} = \mu_2 F_g$$

$$F_{N_2} = F_g$$

$$\tan\theta = 2\mu_2$$

$$\Rightarrow \quad F_{N_1} = \frac{1}{2} F_g \tan\theta .$$

These are exactly the results obtained for N5S.8 with no friction between ladder and wall.

(b) Assume frictionless surface between ladder and floor $\mu_2 = 0$

$$\Rightarrow \quad F_{N_1} = 0$$

$$F_{N_2} = F_g - F_{f_1}$$

$$F_{f_1} \tan\theta = \frac{F_g}{2} \tan\theta$$

$$\Rightarrow \quad F_{f_1} = \frac{F_g}{2}$$

$$\Rightarrow \quad F_{N_2} = F_g - \frac{F_g}{2} = \frac{F_g}{2}$$

$$\tan\theta = \frac{2\mu_2}{1-\mu_1\mu_2} \to 0$$

$\Rightarrow$ ladder starts sliding at the smallest angle because there is no friction between ladder and floor.

Ok this was fun!

It is the complete solution!!.

Moving on ....