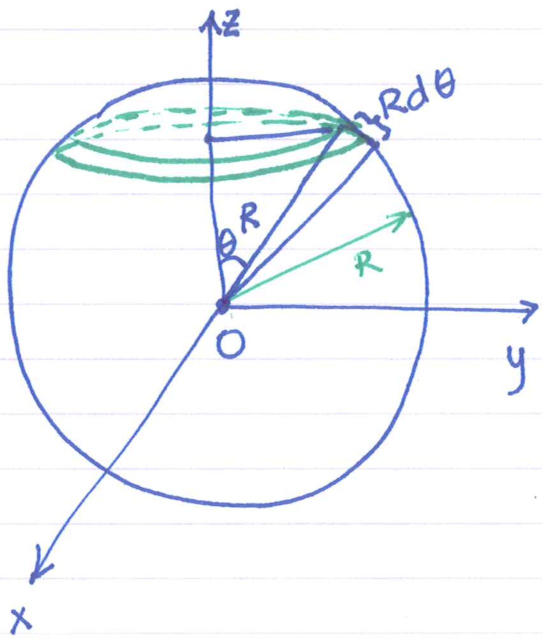
⑤ Spherical Hollow Shell:

Axis of rotation along $\hat{z}$

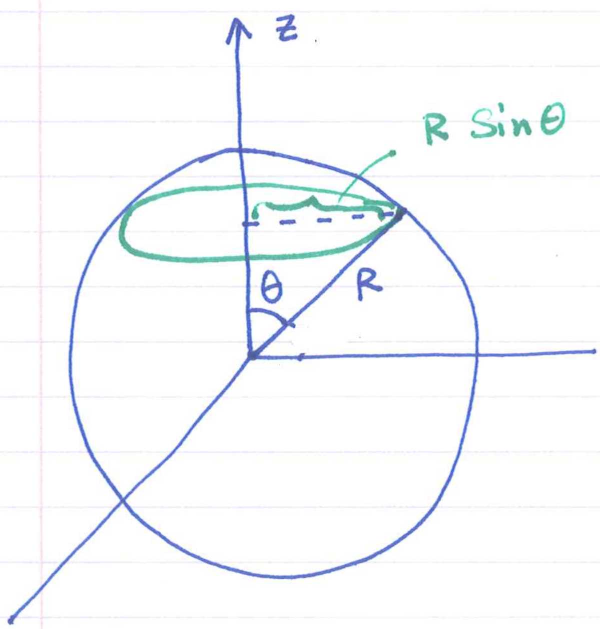
Mass M

Radius R



Slice the spherical shell into hoops perpendicular to the z-axis.

The radius of the hoop varies from zero at the poles to R at the equator



Top view of hoop

Radius of hoop = R Sinθ

$\theta = 0$ at N pole

$\theta = \frac{\pi}{2}$ at equator

$\theta = \pi$ at S pole.

$$\rho = \frac{\text{mass}}{\text{area}} = \frac{M}{\underbrace{4\pi R^2}_{\text{surface area of spherical shell}}}$$

$$I_{hoop} = (dm)\, r^2$$

$$= \rho\, (\text{area of hoop})\, r^2$$

$$= \rho\, (2\pi\, R\sin\theta \cdot R\, d\theta)\, (R\sin\theta)^2$$

$$= 2\pi\, \rho\, R^4 \sin^3\theta\, d\theta$$

$$I_{shell} = \int I_{hoop} = 2\pi\, \rho\, R^4 \underbrace{\int_0^{\pi} \sin^3\theta\, d\theta}_{\frac{4}{3}}$$

$$= 2\pi\, \frac{M}{4\pi R^2}\, R^4\, \frac{4}{3}$$

$$\boxed{I_{shell} = \frac{2}{3} M R^2}$$

Integral of $\int_0^{\pi} \sin^3\theta \, d\theta$

$$= \int_0^{\pi} \underbrace{\sin^2\theta}_{1-\cos^2\theta} \; \underbrace{\sin\theta \, d\theta}_{-d\cos\theta}$$

Let $\cos\theta = x$

$$= -\int_1^{-1} (1-x^2) \, dx$$

use negative sign to switch limits.

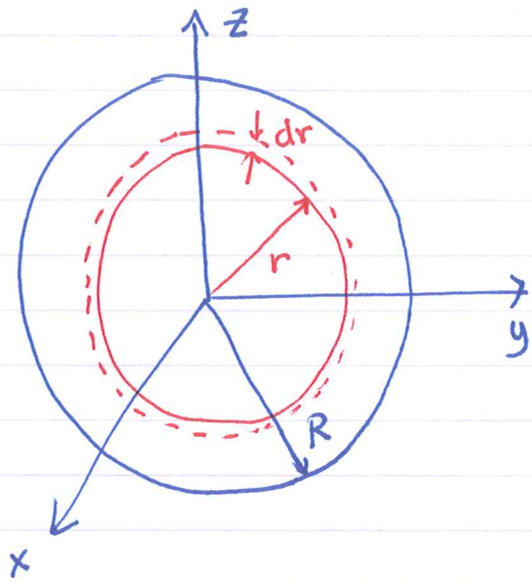
$$= \left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1}$$

$$= 1 - \frac{1}{3} - \left[-1 + \frac{1}{3}\right]$$

$$= \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

$$\boxed{\int_0^{\pi} \sin^3\theta \, d\theta = \frac{4}{3}}$$

⑥ Solid Sphere



Axis of rotation along $\hat{z}$

Mass M

Radius R

There are several ways of slicing and dicing the sphere.

We could slice it into disks.

The algebra is simpler if we decompose the sphere into shells like an onion.

$$\rho = \frac{\text{Mass}}{\text{Volume}} = \frac{M}{\frac{4}{3}\pi R^3}$$

$$I_{shell} = \frac{2}{3}(dm)\, r^2 = \frac{2}{3}\rho\,(4\pi r^2\, dr)\, r^2$$

$$I_{sphere} = \frac{2}{3}\,\frac{M}{\frac{4}{3}\pi R^3}\,4\pi \underbrace{\int_0^R r^4\, dr}_{R^5/5}$$

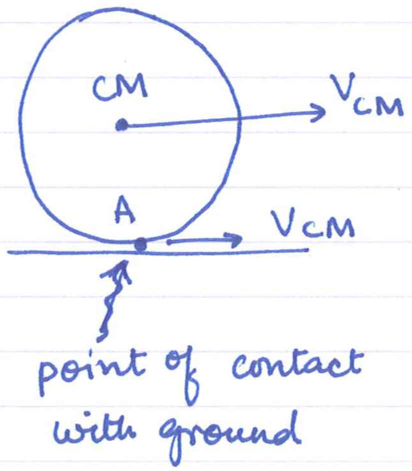
$$\boxed{I_{sphere} = \frac{2}{5} M R^2}$$

Translation & Rotation

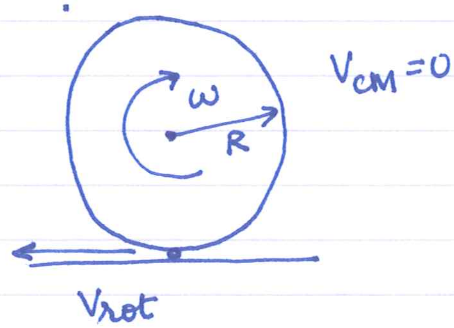
$$K = K^{CM} + K^{rot}$$

$$\boxed{K = \frac{1}{2} m V_{CM}^2 + \frac{1}{2} I \omega^2} \quad -①$$

Only translation

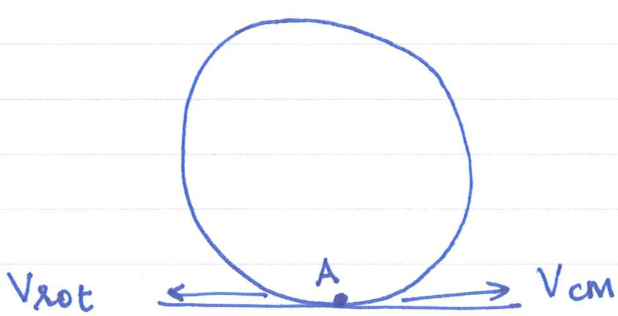


Only rotation



$$V_{rot} = R\omega$$

Rotation & Translation



$$\vec{V}_A = \vec{V}_{CM} + \vec{V}_{rot}$$

$$|\vec{V}_A| = |V_{CM} - R\omega|$$

IF ball rolls without slipping $\Rightarrow \vec{V}_A = 0$

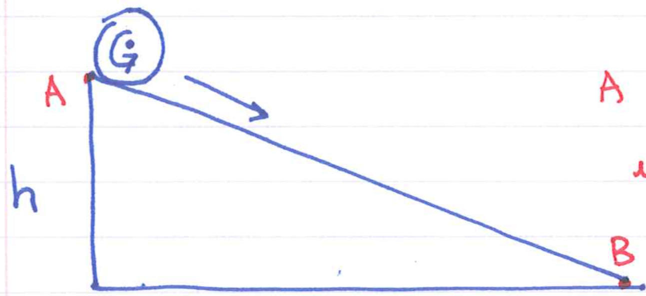
the relative velocity of ball with respect to ground at A is zero.

$$\Rightarrow V_{CM} = R\omega \Rightarrow \boxed{\omega = \frac{V_{CM}}{R}} \quad -②$$

Substitute (2) in (1)

$$K = \frac{1}{2} m V_{CM}^2 + \frac{1}{2} I \left(\frac{V_{CM}}{R}\right)^2$$

$$= \frac{1}{2} m V_{CM}^2 \left[1 + \frac{I}{mR^2}\right]$$



A ball rolls down an incline plane

What is its velocity when it reaches the bottom?

Use total energy conservation :

$$E_A = E_B$$

$$\underbrace{\cancel{K_A}}_{0} + V_A = K_B + \underbrace{\cancel{V_B}}_{0}$$

$$mgh = K_B^{CM} + K_B^{rot} = \frac{1}{2} m V_{cm}^2 \left[1 + \frac{I}{mR^2}\right]$$

$$\Rightarrow \boxed{V_{CM}^2 = \frac{2gh}{\left(1 + \frac{I}{mR^2}\right)}}$$

Moment of Inertia of different Shapes

	I	$I/MR^2 \equiv \alpha$
Hoop	MR^2	1
Disk	$\frac{MR^2}{2}$	$\frac{1}{2}$
Hollow Ball/ Spherical Shell	$\frac{2}{3}MR^2$	$\frac{2}{3}$
Solid Ball / Sphere	$\frac{2}{5}MR^2$	$\frac{2}{5}$

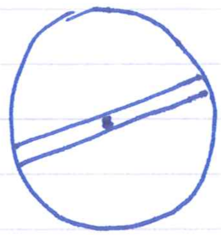
$$\boxed{v_{cm}^2 = \frac{2gh}{1+\alpha}}$$

Check: For $\alpha = 0$ i.e. for $I = 0$ or no rotation

$$v_{cm}^2 = 2gh \quad \checkmark$$

Analyzing the race

(1)



Hoop with a rod across the diameter

M_h = mass of hoop

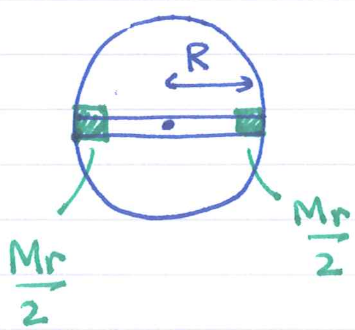
M_r = mass of rod

R = radius of hoop

$$I_1 = I_{hoop} + I_{rod}$$

$$= M_h R^2 + \frac{M_r (2R)^2}{12} = \left(M_h R^2 + \frac{M_r R^2}{3}\right)$$

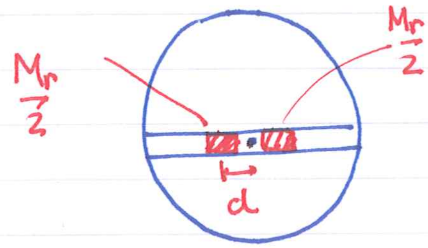
(2)



Hoop with rod but with masses concentrated at the edges

$$I_2 = M_h R^2 + \frac{M_r}{2} R^2 \times 2$$ from the two masses

$$I_2 = M_h R^2 + M_r R^2$$

③ 

Hoop with masses concentrated at a distance d away from the center

$d \ll R$

(We don't want to make $d=0$ otherwise the moment of inertia will be zero)

$$I_3 = M_h R^2 + \frac{M_r}{2} d^2 \times 2$$

$$= M_h R^2 + M_r d^2$$

$d \ll R.$

$$I_3 < I_1 < I_2$$

Which of these objects will reach the ground fastest?

The larger the moment of inertia I the more the rotational kinetic energy K^{rot} of that object. Since $K^{rot} + K^{cm}$ must add up to the same potential energy due to gravity mgh we see that a larger K^{rot} means a smaller K^{cm}

$\Rightarrow$ $\boxed{\text{the object with the larger I has a small } V_{cm}}$ & will take longer to reach the ground.

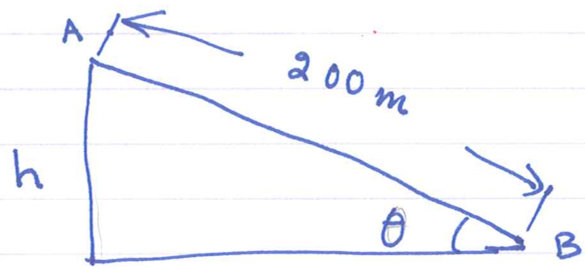
$$V_{CM}^2 = \frac{2gh}{1 + \frac{I}{mR^2}}$$

$m = M_h + M_r$ total mass of object.

and I has been calculated for the three mass distributions so we can get V_{cm} in all cases.

9/14/2012 Problems in class

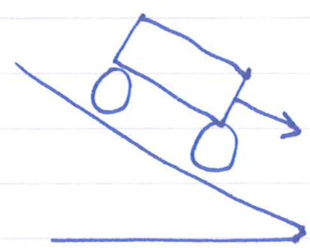
C9R.2



$m_B = 36\ kg$

$m_W = 3\ kg$

r_W



$$E_A = E_B$$

$$Mgh = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I \omega^2$$

$$M = m_B + 4 m_W$$

$$I = \alpha\, m_W r_W^2 \times 4$$

↑ 4 wheels

$\alpha = \frac{1}{2}$ (solid wheel)

$= 1$ (bicycle wheel)

no-slip condition

$$\omega = \frac{v_{cm}}{r_W}$$

$$Mgh = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} \alpha\, m \cancel{r_W^2} \times 4 \frac{v_{cm}^2}{\cancel{r_W^2}}$$

$$2Mgh = v_{cm}^2 (M + 2\alpha m)$$

$$v_{cm}^2 = \left[\frac{2gh}{1 + 2\alpha m/M} \right]$$

$$v_{cM}^{b} = \sqrt{\frac{2gh}{1 + 2\,m/M}}$$

$$v_{cm}^{d} = \sqrt{\frac{2gh}{1 + m/M}}$$

time to travel down incline $t = \dfrac{\text{distance}}{v_{cm}}$

$$t^{b} = \sqrt{\frac{1 + 2\,m/M}{2gh}} \times d$$

$$t^{d} = \sqrt{\frac{1 + m/M}{2gh}} \times d$$

$t^{b} > t^{d}$ <u>bicycle wheel is slower</u>

$t^{d} = 52$ sec solve for h

Calculate t^{b}

$$\frac{t^{b}}{t^{d}} = \sqrt{\frac{1 + 2\,m/M}{1 + m/M}} = \sqrt{\frac{1 + 2\,\frac{3}{\cancel{36}\,48}}{1 + 3/\cancel{36}\,48}} = \sqrt{\frac{1 + \frac{1}{\cancel{6}8}}{1 + \frac{1}{\cancel{12}\,16}}}$$

$$= \cancel{\sqrt{\frac{7/6}{13/12}} = \sqrt{\frac{14}{13}}} \qquad \frac{9 \times \cancel{16}\,2}{8 \times 17} = \sqrt{\frac{18}{17}}$$

$$Mgh = \frac{1}{2} M V_A^2 + 4\left(\frac{1}{2}\right)\left(\frac{1}{2} m R^2\right)\left(\frac{V_A}{R}\right)^2$$

$$= \frac{1}{2} M V_B^2 + 4 \frac{1}{2} \left(m R^2 \right)\left(\frac{V_B}{R}\right)^2$$

$$Mgh = \frac{1}{2} M V_A^2 + m V_A^2 = V_A^2 \left(\frac{1}{2} M + m\right)$$

$$= \frac{1}{2} V_A^2 (M + 2m)$$

$$= \frac{1}{2} M V_B^2 + 2 m V_B^2$$

$$= \frac{1}{2} V_B^2 (M + 4m)$$

$$2gh = V_A^2 \left(1 + 2\frac{m}{M}\right) = V_B^2 \left(1 + 4\frac{m}{M}\right)$$

$$\frac{V_A^2}{V_B^2} = \left(\frac{1 + 4\, m/M}{1 + 2m/M}\right) = 1 + 4\ 3/48$$